King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMFM06 NUMERICAL AND COMPUTATIONAL METHODS IN FINANCE

AUGUST 2017

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS.

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS.

IF MORE THAN FOUR QUESTIONS ARE ATTEMPTED, THEN ONLY THE BEST FOUR WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. An investor models 10 different stocks as each having a price $(S_i)_t$ (with $1 \leq i \leq 10$) which follows geometric Brownian motion

$$\mathrm{d}(S_i)_t = \mu_i \,\mathrm{d}t + \sigma_i \,\mathrm{d}(W_i)_t$$

for some constants μ_i and σ_i. The $(W_i)_t$ are correlated Brownian motions with correlation matrix P.

(i) Write down the stochastic differential equations for the logarithm of each stock price and hence express $(S_i)_t$ in terms of $(W_i)_t$. [**20%**]

(ii) Define the *Cholesky decomposition* and explain how you could use this to simulate $(W_i)_T$ at a fixed time T and hence $(S_i)_T$. [**30%**]

(iii) The investor has an amount A which they will invest in the different stocks at time 0. They will then hold the stocks till time T at which point they will calculate the value v of the portfolio. The investor associates the utility $1 - \exp(-v)$ to this value. The investor must invest all of this amount in stocks and short selling is prohibited. Write down a mathematical formulation of the optimization problem of choosing the portfolio which gives the investor the optimum expected utility. [**20%**]

(iv) How could you use a simulation of the $(S_i)_t$ to estimate the expected utility? [**10%**]

(v) Explain briefly how you could use such a simulation to estimate the optimal portfolio. [**20%**]

2. (i) In the market given by the Black–Scholes model, suppose that trader sells a call option and hedges the risk by following a discrete-time delta-hedging strategy at N evenly-spaced time intervals up to maturity. Explain what cashflows occur at each time point and hence write down difference equations for the trader's bank balance at each time point. [**40%**]

(ii) Describe mathematically how you could simulate the stock price at each time point in order to test the effectiveness of this strategy. [**20%**]

(iii) The profit and loss for a trader pursuing this strategy will be a random variable whose variance depends upon N, the number of time intervals. Sketch a log-log plot of this variance against N, describing any important features of your graph. [**10%**]

(iv) Suppose that the stock price does not follow the Black–Scholes model but instead there is a bid-ask spread. The ask price is given by a process S_t following geometric Brownian motion, but the bid price is given by $(1-\epsilon)S_t$ for some $\epsilon > 0$. How would the difference equations for the cashflows change? [**20%**]

(v) How would you expect the log-log plot to change? [**10%**]

3. (i) You wish to compute the price of the following options using the explicit finite difference method applied to the Black–Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

What boundary conditions would you use in each case?

(a) A European call option with strike K.

(b) An American put option with strike K.

(c) A down and out knock-out call option with strike K and barrier $B < K$.

[**60%**]

(ii) How could you price a knock-in call option with the same strike and barrier as the knock-out call option? [**10%**]

(iii) Derive the difference equations for the explicit finite difference scheme to price a European call option when $r = 0$. [**30%**]

4. In a local volatility model, all the assumptions of the Black–Scholes model hold except that the stochastic differential equation followed by the stock price S_t is:

$$\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma(S,t)\,\mathrm{d}W_t)$$

where $\sigma(S,t)$ is a given function of S and t. μ is a constant. You may assume that $\sigma(S,t)$ is smooth and bounded both above and below.

(i) What is the Euler scheme for the stock price? [**20%**]

(ii) Describe how you could price a European call option in this model using the Monte Carlo method. [**20%**]

(iii) How would you estimate the error in this price? [**10%**]

(iv) How could you use the control variate method to improve your estimate. [**30%**]

(v) When using the Euler scheme to simulate stock prices in this way, the simulated prices may sometimes be negative. For some purposes this will be undesirable as stock prices are never negative. Suggest a way to simulate the stock prices which does not suffer from this problem. [**20%**]

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5. (i) What is meant by the implied volatility of a European call option? [**10%**]

(ii) Suppose that you have written a MATLAB function

```
bsPrice(K,T,r,S,sigma)
```

which computes the Black–Scholes price of a call option given its strike, K, maturity, T, and the market data of the risk-free interest rate r, the current stock price S and the stock volatility σ. Write a MATLAB function to compute the implied volatility of an option. [**30%**]

(iii) What is meant by the volatility smile? What does the existence of a volatility smile tell us about the Black–Scholes model? [**20%**]

(iv) What does it mean to calibrate a stock price model to option prices? [**10%**]

(v) Suppose you are given a function $\mathrm{HP}(K, T, r, S, \sigma_0, \xi, \kappa, \theta\rho)$ which computes the price of a European call option according to the Heston model. The variables σ_0, ξ, κ, θ and ρ are additional parameters required by the Heston model and they must all lie in some given set $X \subseteq \mathrm{R}^4$. Suppose that there are n European call options traded in the market with strikes K_i, maturities T_i and prices P_i. Write down a mathematical formulation of a minimization problem you could solve to calibrate the Heston model to the market data. [**20%**]

(vi) What problems may occur if one attempts to solve this minimization problem using MATLAB's `fmincon` function? [**10%**]