King's College London

UNIVERSITY OF LONDON

PLACE THIS PAPER AND ANY ANSWER BOOKLETS **in the EXAM ENVELOPE provided**
Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMFM06 NUMERICAL AND COMPUTATIONAL METHODS IN FINANCE

MAY 2019

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS. FULL MARKS ARE AWARDED FOR CORRECT ANSWERS TO FOUR QUESTIONS. IF MORE THAN FOUR ARE ATTEMPTED THEN ONLY THE BEST FOUR WILL COUNT.

YOU ARE PERMITTED TO USE A CALCULATOR.
ONLY CALCULATORS FROM THE CASIO FX83 AND FX85 RANGE ARE ALLOWED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (a) What is meant by a European knock-out call option with strike K, maturity T, and barrier B on a stock? You should write a mathematical formula giving the payout of the option in terms of the stock price S_t at times $t \in [0, T]$. [**20%**]

(b) Suppose that the market follows the Black–Scholes model: in particular with the stock following geometric Brownian motion with drift μ and volatility σ; zero coupon bonds are available at a risk-free rate of r. Write the MATLAB code to simulate the stock price paths. [**20%**]

(c) Describe how you could use such a simulation to estimate the price of a European Knock-Out option by the Monte Carlo method. [**20%**]

(d) How would you estimate the Delta and Gamma of a European Knock-Out option using the Monte Carlo method? [**20%**]

(e) Why might a trader want to know the Gamma of the option? [**10%**]

(f) Explain why the same Monte Carlo method cannot be used to price an American Knock-Out option. What could you do instead to price an American Knock-Out option? [**10%**]

2. Suppose that a stock price process, S_t, follows the SDE

$$\mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1)$$

where W_t^1 is a Brownian motion. Suppose that the short-term interest rate r_t follows the SDE

$$\mathrm{d}r_t = a(r_t - b)\mathrm{d}t + c\,\mathrm{d}W_t^2$$

for some constants a, b and c, and where W_t^2 is a Brownian motion independent of W_t^1. You wish to simulate the processes S_t and r_t at discrete times 0, δt, $2\,\delta t$, $\ldots$, $i\,\delta t, \ldots, N\,\delta t$.

(a) Write the difference equation you would use to simulate r_t using the Euler scheme. [**20%**]

(b) Explain mathematically how you would simulate S_t, explaining why you would not use the Euler scheme for S_t. [**30%**]

(c) Suppose that a trader incorrectly believes that the interest rate is a constant $r = r_0$. They write a call option with strike K and maturity $T = N\,\delta t$ for the Black–Scholes price P, and then follow the Black–Scholes delta hedging strategy in discrete time. Write down difference equations that you could use to compute their bank balance b_t at time $t = i\,\delta t$. You should assume that over the time interval from $i\,\delta t$ to $(i+1)\delta t$ their bank balance grows at the continuously compounded rate r_t. [**30%**]

(d) Suppose that W_t^1 and W_t^2 are not independent, but are instead correlated with correlation ρ. How would you change your simulation to account for this? [**20%**]

3. (a) What is meant by the Value at Risk of a portfolio? [**20%**]

(b) What is meant by the Expected Shortfall of a portfolio? [**10%**]

(c) If a portfolio has current value P and its value at time T is normally distributed with mean M and standard deviation σ write down a formula to compute the Value at Risk of this portfolio over time horizon T in terms of N^{-1}, the inverse cumulative distribution function of the standard normal distribution. [**20%**]

(d) Suppose that the market contains n assets. The current price of asset i $(1 \leq i \leq n)$ is P_i. At time T, the asset prices follow a multivariate normal distribution with means M_i and with covariance matrix Ω. Give a formula for the standard deviation of a portfolio consisting of α_i units of each asset i at time T. [**10%**]

(e) An investor has a budget B and wishes to invest this in a portfolio of the n assets in such a way as to make an expected profit of Q at time T while minimizing the portfolio's Value at Risk at time T at confidence level 95%. Give a mathematical formulation of this problem, justifying your answer, and explain how you would solve it in MATLAB. [**30%**]

(f) How would you find a portfolio that minimizes the Expected Shortfall rather than the value at risk? [**10%**]

4. (a) Let S be a subset of $[-1,1]^3$. One way to approximate the volume of S is by counting cubes. We divide each interval $[-1,1]$ into n subintervals and so divide the cube $[-1,1]^3$ into n^3 smaller cubes. One can then compute the proportion of cubes whose center lies in S and hence estimate the volume of S. Write MATLAB code to implement this algorithm in the case where S is given by

$$S = \{(x,y,z) \mid x^4 + y^4 + z^4 \leq 1.\}$$

[**50%**]

(b) How would you test your code? [**10%**]

(c) For subsets, S of $[-1,1]^3$ with smooth boundary, the error in this algorithm is proportional to $O(n^{-1})$. Describe how you could test this fact numerically, being sure to describe how you would interpret the results. [**20%**]

(d) Describe briefly how you could compute the volume of S by a Monte Carlo method. [**10%**]

(e) Which is a more efficient algorithm: counting cubes or the Monte Carlo method? Justify your answer. [**10%**]

5. (a) The Black–Scholes PDE is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Describe the change of variables is required to transform this to the backwards heat equation

$$\frac{\partial U}{\partial t} = -\frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} \quad .$$

Justify your answer. [**30%**]

(b) A digital call option of strike K which has payoff at maturity time T given by

$$\text{payoff} = \begin{cases} 1 & \text{when } S_T \geq K \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that you wish to price this derivative by applying the implicit finite difference method to the backwards heat equation.

(i) Over what region of the (t, x)-plane would you perform the calculation? Justify your answer. [**20%**]

(ii) What boundary conditions would you use? [**20%**]

(iii) Derive the difference equations used for this finite difference method at points away from the boundary. [**30%**]