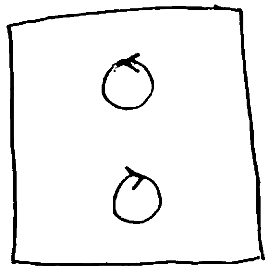
# Solutions 1

1. See Donaldson, Ch 2 Lemma 1.

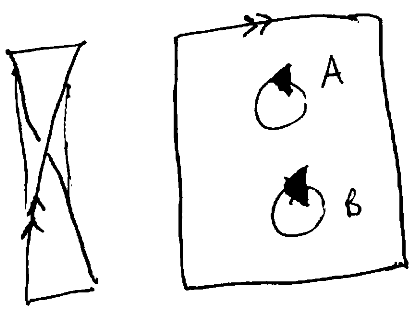
An alternative proof is given below.

We know that the Klein bottle $K = \mathbb{R}P^2 \# \mathbb{R}P^2$

The Klein bottle with a disc removed is equivalent to the space obtained by gluing the two circles below with the identification given by the arrows

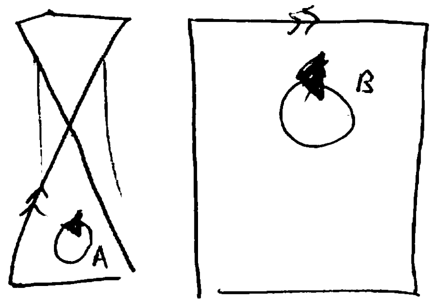
$K \setminus D \cong$ 

So $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ is given by gluing a Möbius strip onto the boundary of $K \setminus D$

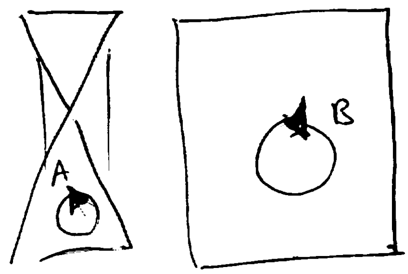
$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \cong$ 


By moving circle A onto the Möbius strip then sending it round once, we get

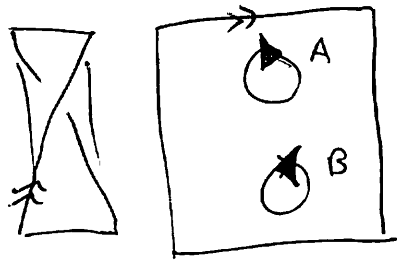
$\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \cong$



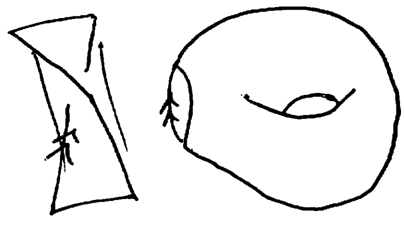

$\cong$




$\cong$




$\cong$



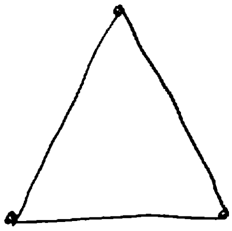
$\cong \mathbb{RP}^2 \# T^2$

If $g \geq 0$ and $h > 0$

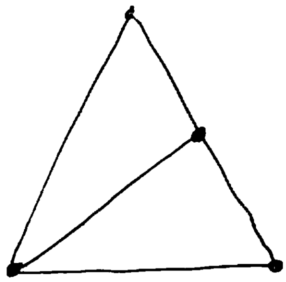
$$\therefore \Sigma_g \# \Xi_h \cong \underbrace{T^2 \# T^2 \# \dots \# T^2}_{g \text{ times}} \# \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{h \text{ times}}$$

$$\cong \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{h+2(g) \text{ times}} \cong \Xi_{h+2(g)}$$

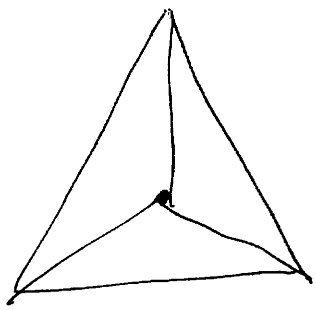
2. Euler characteristic = V − E + F



→ Euler characteristic 1



→ Euler characteristic 1



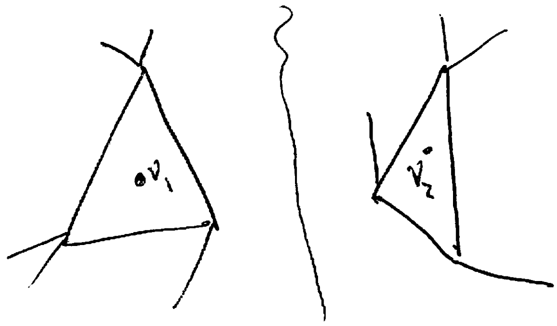
→ Euler characteristic 1

So given a triangulation of a surface, further subdivisions of the triangulation won't change the Euler characteristic.

Given two triangulations, find a third triangulation which is a refinement of both by progressively refining the first triangulation (you should perturb the initial two triangulations to ensure they are transverse first).

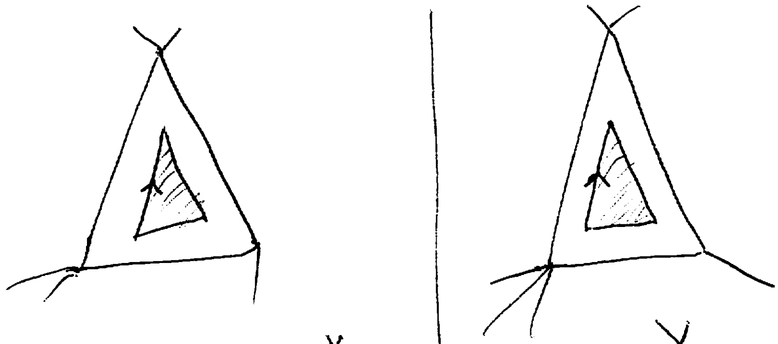
This gives an extremely hand-wavy proof that both triangulations have the same Euler characteristic.

3. Pick a triangulation for $X$ and a triangulation for $Y$. Pick points $v_1, v_2$ in the centres of two faces of the triangulation



X Y

Forming the connect sum by gluing neighbourhoods of $v_1$ and $v_2$ is equivalent to cutting out the shaded area, identifying the triangles shown below



X Y

We can refine the triangulations of $X$ and $Y$ to include the vertices and edges we are ~~cutting~~ identifying. This gives a triangulation for $X \# Y$ with 3 less vertices, 3 less edges and two less faces than our triangulation for $X \amalg Y$.

So $$\chi(X \# Y) = \chi(X) + \chi(Y) - 3 + 3 - 2$$
$$= \chi(X) + \chi(Y) - 2$$

So $$\chi(\Sigma_g) = 2 - 2g$$
$$\chi(\ominus_h) = 2 - h$$

4. This is meant to have you scratching your head!

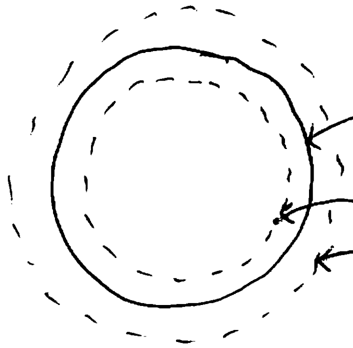
Start with:

$$(x^2+y^2-1)^2+z^2=0$$

This gives the equation of a circle in $\mathbb{R}^3$

$$(x^2+y^2-1)^2+z^2=\varepsilon$$

"thickens" this to give a torus if $\varepsilon$ is small. To see why, plot the contours of $(x^2+y^2-1)^2$



Contours $(x^2+y^2-1)^2=0$

Contours $(x^2+y^2-1)^2=\alpha^2$

$\Longleftrightarrow x^2+y^2-1=\pm\alpha$

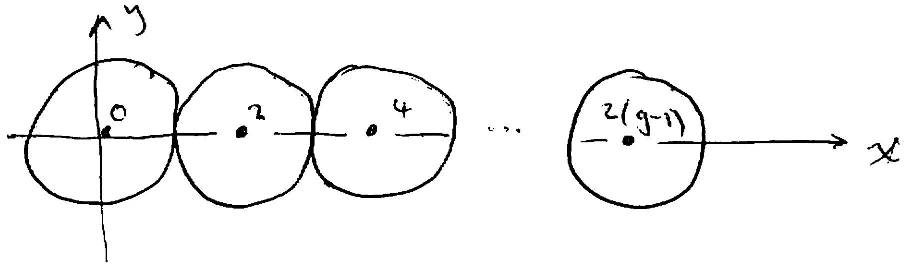
(Notice that this doesn't give the "round" torus. Its easy to see that the standard torus in $\mathbb{R}^3$ is given by $(x^2+y^2+z^2+R^2-r^2)^2 - 4R^2(x^2+y^2)$

consult Wikipedia under "Torus")

Now consider:

$$\prod_{n=0}^{g-1} \left( (x-2n)^2 + y^2 \right)^2 + z^2$$

The zero set of this consists of circles:




Which we again thicken to get a genus $g$ surface

$$\prod_{n=0}^{g-1} \left( (x-2n)^2 + y^2 \right)^2 + z^2 = \varepsilon$$

for small $\varepsilon$.

# A MATHEMATICA EXAMPLE

```
In[47]:= ContourPlot3D[(x^2 + y^2 - 1)^2*((x - 2)^2 + y^2 - 1)^2 + z^2 == 1, {x, -2, 6}, {y, -4, 4
   {z, -4, 4}, PlotPoints->100]
```

Out[47]=