Almost-Kähler Geometry

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Abstract

The central theme of this thesis is almost-Kähler, Einstein 4-manifolds.

We shall show how to answer the question of when a given Riemannian metric admits a compatible almost-Kähler structure. We shall illustrate this by proving that hyperbolic space of any dimension does not admit a compatible almost-Kähler structure. In the case of four dimensional manifolds, we shall show further that no anti-self-dual, Einstein manifold admits a non-Kähler, almost-Kähler structure. Indeed we shall prove that any Einstein, weakly $*$-Einstein, almost-Kähler 4-manifold is given by a special case of the Gibbons–Hawking ansatz.

We shall consider a number of other curvature conditions one can impose on the curvature of an almost-Kähler 4-manifold. In particular we shall show that a compact, Einstein, almost-Kähler 4-manifold whose fundamental two form is a root of the Weyl tensor is necessarily Kähler. We shall also show that a compact, almost-Kähler, Einstein 4-manifold with constant $*$-scalar curvature is necessarily Kähler.

We shall prove, using the Seiberg–Witten invariants, that rational surfaces cannot admit a non-Kähler almost-Kähler, Einstein structure.

We shall also briefly consider the related topic of Hermitian, Einstein 4-manifolds. We find a new proof of the relationship between Ricci-flat Hermitian manifolds given in [PB87] and the SU(∞)-Toda field equation and obtain an analogous result for Hermitian, Einstein manifolds with non-zero scalar curvature.

Introduction

The initial motivation for this thesis comes from the following conjecture due to Goldberg:

Conjecture 1 *[Gol69] A compact, almost-Kähler, Einstein manifold is necessarily Kähler.*

An almost-Kähler manifold is an almost-Hermitian manifold whose fundamental two form is closed. In other words, an almost-Kähler manifold is a symplectic manifold equipped with a compatible metric. Broadly speaking, this thesis is about what the possibilities are for the curvature of an almost-Kähler manifold.

Goldberg's conjecture is still far from resolved — and we shall not resolve it. The conjecture will, however, serve as a guide throughout this thesis.

One natural approach to Goldberg's conjecture is to impose additional curvature conditions and see whether or not one can prove that almost-Kähler manifolds which satisfy these additional conditions are necessarily Kähler. Of course, the motivation for studying such problems is two-fold: one may obtain some insight into how to prove non-existence results for almost-Kähler, Einstein metrics; alternatively one may obtain some insight into how to construct such metrics. Experience suggests that the Einstein metrics which are easiest to find are the ones which satisfy the most curvature conditions.

We recall that the curvature decomposes into three pieces under $\mathrm{SO}(2n)$ — namely the trace free Ricci tensor, the Weyl tensor and the scalar curvature. When $n \geq 3$ all of these components are irreducible. However, when $n = 2$ (i.e. when the manifold is four dimensional), the Weyl tensor decomposes into the self-dual and anti-self-dual parts of the Weyl tensor.

On an Einstein manifold, the trace free part of the Ricci tensor vanishes and the scalar curvature is constant. Thus the only interesting part of the curvature tensor is the Weyl tensor. Thus any additional curvature conditions we wish to impose will be conditions on the Weyl tensor of our manifold. Of course, this immediately makes the four-dimensional case stand out. Most of this thesis will be devoted to the four-dimensional case.

The self-dual part of the Weyl tensor decomposes under U(2) into three pieces, one of which is a scalar. The anti-self-dual part of the Weyl tensor remains irreducible under U(2). So the curvature tensor of an Einstein, almost-Hermitian 4-manifold has 4 interesting components. Correspondingly there are 4 special types of almost-Hermitian, Einstein 4-manifold each imposing one additional condition on the Weyl tensor:

1. weakly $*$-Einstein,
2. constant $*$-scalar curvature,
3. $W^+_{00} \equiv 0$,
4. self-dual.

The names are, admittedly, not memorable. In this introduction we shall refer to these conditions simply as conditions (1), (2), (3) and (4). Manifolds with some, or all, of the above curvature properties constitute the most natural types of special almost-Hermitian, Einstein 4-manifold. Even if we cannot prove the Goldberg conjecture, we would like to prove that compact, almost-Kähler, Einstein manifolds with some of these additional properties are necessarily Kähler. We should point out that all four conditions are defined for non-Einstein almost-Hermitian manifolds. However, in this case conditions (1) and (2) are no longer conditions on the Weyl tensor alone.

Before the author began this thesis, a certain amount of work had been done on such questions. One important result was the proof ([Sek85] and [Sek87]) that, if one makes the additional assumption that the manifold has positive scalar curvature, then Goldberg's conjecture is true. Nevertheless, little was known about the case of negative scalar curvature. Indeed the following question was still unanswered:

Question 1 *Can a constant-curvature manifold admit a compatible strictly almost-Kähler structure?*

This question was answered by Olzsak in [Ols78] for manifolds of dimensions 8 and above, but the problem was still unresolved in dimensions 4 and 6 — although Sekigawa and Oguro proved in [SO94] that the result is true if one looks for a global almost-Kähler structure on a complete hyperbolic manifold.

Our first result result towards answering this type of question is given in Chapter 2: we prove that a compact, almost-Kähler, Einstein 4-manifold which satisfying (2) is necessarily Kähler. This tells us that compact anti-self-dual, Einstein, almost-Kähler four manifolds are necessarily Kähler.

However, in terms of what it tells us about the constant-curvature case, this result is unsatisfactory. The condition that the manifold is constant-curvature is about as stringent a condition as one could impose on a manifold's curvature. It tells us that the manifold must be locally isometric to either a sphere or a hyperbolic space or it must be flat. One feels that one should surely be able to prove, using an entirely local argument, that constant-curvature manifolds cannot admit compatible strictly almost-Kähler structures. Our problem is that we have no method of determining when a given Riemannian metric admits a compatible almost-Kähler structure. We shall devise a strategy for answering this question which we shall apply to prove that constant-curvature manifolds cannot admit a compatible strictly almost-Kähler structure. The strategy is rather complex in that it requires examining a surprisingly large number of derivatives of the curvature. So we shall have to build up a good body of knowledge about the curvature, and derivatives of the curvature, of almost-Hermitian manifolds.

However, once we have done this, we shall be able to use our strategy to generalise our result. It will be a relatively simple matter to prove that an almost-Kähler, Einstein 4-manifold satisfying either both (1) and (2) or both (1) and (3) must be Kähler — even locally. This includes anti-self-dual, Einstein manifolds.

Another line of enquiry one might consider in studying Goldberg's conjecture is whether or not almost-Kähler, Einstein metrics exist locally. Once again this question was unanswered when the author began his thesis. The author attempted to tackle the problem by applying Cartan–Kähler theory to obtain an abstract existence result. Indeed, he believed he had succeeded in doing so. Nevertheless a gap in the proof was pointed out to the author which seems too difficult to fix. Notice that experts on Cartan–Kähler theory had believed that it would be easy to prove the existence of almost-Kähler, Einstein metrics using Cartan–Kähler theory. Thus the difficulty one experiences in attempting to do this is surprising.

Between the time of producing the "proof" and the mistake being pointed out, Nurowski and Przanowski ([PN]) found an explicit example of an almost-Kähler, Einstein metric. Motivated by their example, Tod ([Tod97a]) went on to find a family of examples all based on the Gibbons–Hawking ansatz. We shall refer to this method of producing almost-Kähler, Einstein metrics as Tod's construction.

Since explicit examples now exist, the motivation for providing an abstract proof of the existence of almost-Kähler Einstein metrics has rather diminished. Nevertheless, Tod's examples raise as many questions as they answer. For example: do there exist almost-Kähler Einstein metrics in 4-dimensions which are not given by Tod's construction? Additional motivation for asking such a question is given by the fact that a comparable result is true for Hermitian, Einstein manifolds. Specifically one has the *Riemannian Goldberg–Sachs* theorem which states that any Hermitian Einstein manifold automatically satisfies curvature condition (1).

There is a sense in which Cartan–Kähler theory should allow one to answer such questions as "Is there any unexpected condition the curvature of an almost-Kähler, Einstein manifold must satisfy?". If one could provide a proof using Cartan–Kähler theory for the existence of almost-Kähler, Einstein metrics then one would be able to answer such a question as an immediate corollary.

Thus, it is still natural to attempt to apply Cartan–Kähler theory to our problem and so we shall do this in chapter 4. Our results are incomplete but we do find one interesting result — specifically an unexpected condition that the curvature and its first derivative must satisfy on an almost-Kähler, Einstein 4-manifold. We shall use this to prove that compact, almost-Kähler, Einstein 4-manifolds whose curvature satisfies condition (3) are necessarily Kähler.

One other question that Tod's construction raises is whether or not we can find a neat categorisation of the metrics it produces. We are able to answer this problem by combining our strategy for finding out whether or not a metric admits a compatible almost-Kähler structure with some observations on the geometry of Tod's examples. The result is that any strictly almost-Kähler, Einstein 4-manifold whose curvature satisfies condition (1) must be given by Tod's construction. An immediate corollary of this is that such manifolds cannot be compact.

We can summarize our findings on the result of adding additional curvature conditions to Goldberg's conjecture as follows:

Theorem 1 *A compact almost-Kähler, Einstein* 4*-manifold which satisfies one of (1), (2) or (3) is necessarily Kähler.*

A non-compact almost-Kähler, Einstein 4*-manifold which in addition satisfies (1) is given by Tod's construction.*

By way of contrast, Hermitian manifolds always satisfy (3). The Riemannian Goldberg–Sachs theorem tells us that Hermitian Einstein manifolds satisfy (1) and (3). Kähler, Einstein manifolds always satisfy (1), (2) and (3). In this sense our results give both necessary and sufficient conditions for a compact almost-Kähler, Einstein manifold to be Kähler.

There are, of course, other approaches to Goldberg's conjecture. Instead of strengthening the curvature conditions one might want to strengthen the topological conditions. It is easy to find topological obstructions to the existence of almost-Kähler, Einstein metrics by simply combining known obstructions to the existence of symplectic structures with known obstructions to the existence of Einstein metrics. When the author began this thesis all known obstructions to the existence of almost-Kähler, Einstein metrics were of this form and so lay outside the remit of almost-Kähler geometry. Nevertheless, the new results of Taubes and LeBrun on the Seiberg–Witten invariants looked like a promising source of topological obstructions to the existence of almost-Kähler, Einstein metrics. This hope was one major motivation for the author beginning his study of almost-Kähler Einstein metrics. We shall be able to prove using the Seiberg–Witten invariants that blow ups of rational and ruled manifolds cannot admit strictly almost-Kähler, Einstein metrics even though some of them can admit Einstein metrics and all of them admit symplectic forms.

Finally, we shall also consider some subjects which do not relate so immediately to Goldberg's conjecture but which are of motivational importance.

In particular we shall discuss almost-Kähler, self-dual manifolds, almost-Kähler, anti-self-dual manifolds and Hermitian Einstein manifolds. Since these topics are not of immediate relevance to Goldberg's conjecture, our discussion will be brief.

Chapter 1 is an introduction to almost-Hermitian geometry. Our main aim is to introduce the torsion tensor and the curvature tensor and give some examples of how they interact. We emphasise strongly the importance of the representation theory of $\mathrm{U}(n)$. We shall describe some interesting examples of almost-Hermitian manifolds. We shall examine Tod's examples in detail. We shall also examine a construction of Einstein metrics due to Bérard Bergery which involves almost-Kähler manifolds in an interesting way and thereby further motivates their study. Most of this chapter is introductory. Sections 1.1.4, 1.2.3 are new as is the discussion of the geometry of Tod's examples in Section 1.3.1.

Chapter 2 contains a discussion of the features that make the 4-dimensional case particularly interesting. We discuss self-duality, spinors, integral formulae and the Seiberg–Witten invariants. We prove that compact almost-Kähler, Einstein manifolds satisfying (2) are necessarily Kähler. We give examples of manifolds which cannot admit almost-Kähler, Einstein metrics for topological reasons. The discussion of self-dual and anti-self-dual almost-Kähler manifolds in Section 2.1.4 is new as is the material in Section 2.2.2. The examples of manifolds which cannot admit almost-Kähler Einstein metrics given in Section 2.3.3 are also new.

Chapter 3 describes the proof that all almost-Kähler, Einstein manifolds satisfying (1) are given by Tod's construction. We start by describing the strategy used to prove that hyperbolic space cannot admit an almost-Kähler structure, then illustrate the strategy by proving that anti-self-dual, Einstein metrics cannot admit a compatible almost-Kähler structure. We then prove our categorization of Tod's examples. Finally we illustrate our method's applicability to higher dimensions by proving that hyperbolic space of any dimension cannot admit a compatible almost-Kähler structure. All the material is new except in Section 3.5.2 where we give a new proof of a result due to Olszak.

Chapter 4 describes Cartan-Kähler theory and the insight it allows us in the study of almost-Kähler, Einstein manifolds and similar geometric problems. We begin by giving a brief sketch of Cartan-Kähler theory, we then examine in Section 4.3 what it tells us about almost-Kähler, Einstein manifolds.

The material in Section 4.3 is new. We finish the chapter by explaining how Cartan–Kähler theory helps one understand the Riemannian–Goldberg Sachs theorem and by showing how this in turn leads to a rather complete understanding of the local geometry of Hermitian, Einstein manifolds. In the case of Ricci flat, Hermitian manifolds, we show that such manifolds are determined by the SU(∞)-Toda field equation. This repeats a result due to Przanowski and Bialecki [PB87], however, our derivation is simpler in that it avoids Lie–Bäcklund transformations. We also derive an analogous result for Hermitian, Einstein manifolds with non-zero scalar curvature which is new.

Contents

Chapter 1

Curvature and torsion

1.1 Almost-Hermitian manifolds

1.1.1 Definitions

Let M^{2n} be an even dimensional manifold equipped with a Riemannian metric g and an almost-complex structure J (that is a smoothly varying endomorphism of the tangent bundle with $J^2 = -1$). We say that J and g are compatible if:

$$g(X,Y) = g(JX,JY) \qquad \forall\, X,Y \in TM.$$

A compatible g and J are called an almost-Hermitian structure. More conceptually, an almost-Hermitian structure on a manifold is a reduction of the structure group of the tangent bundle TM from $\mathrm{GL}(2n,\mathbb{R})$ to $\mathrm{U}(n)$.

On any almost-Hermitian manifold we can define the *fundamental two form* $\omega \in \bigwedge^2$ by

$$\omega(X,Y) = g(JX,Y) \qquad \forall\, X,Y \in TM.$$

This two form will be non-degenerate — i.e. we shall have that $\omega^n \neq 0$. A non-degenerate two form ω and an almost complex structure J are said to be compatible if

$$\omega(X,Y) = \omega(JX,JY) \qquad \forall\, X,Y \in TM,$$

in which case we can define a Riemannian metric g by $g(X,Y) = -\omega(JX,Y)$. Similarly a non-degenerate two form ω and a metric g are said to be compatible if $\phi(\omega)^2 = -1$ where $\phi : T^*M \otimes T^*M \longrightarrow \mathrm{End}(TM)$ is the isomorphism defined by the metric. In this case we define $J = \phi(\omega)$. Thus we see that an almost-Hermitian structure could have also been defined either as a metric and compatible non-degenerate two form or as an almost-complex structure and compatible non-degenerate two form.

Note that ω^n will always define an orientation on any manifold with a non-degenerate two form. Since $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{GL}^+(2n,\mathbb{R})$, any almost-complex manifold has a natural orientation. On an almost-Hermitian manifold these orientations coincide.

The simplest example is of course $\mathbb{R}^{2n}$ equipped with the metric

$$g^{can} = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + \ldots + (\mathrm{d}x_{2n})^2,$$

the almost-complex structure

$$J^{can} = \mathrm{d}x_1 \otimes \frac{\partial}{\partial x_2} - \mathrm{d}x_2 \otimes \frac{\partial}{\partial x_2} + \ldots + \mathrm{d}x_{2n-1} \otimes \frac{\partial}{\partial x_{2n}} - \mathrm{d}x_{2n} \otimes \frac{\partial}{\partial x_{2n-1}},$$

and the non-degenerate two form

$$\omega^{can} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 + \ldots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}.$$

Thus we have natural integrability conditions for each one of our structures g, J, ω — namely that they are locally isomorphic to g^{can}, J^{can} or ω^{can} respectively. If g is locally isomorphic to g^{can} then we must have that the Riemann curvature tensor R is zero — and of course conversely if $R \equiv 0$ then g is flat.

Similarly if ω is locally isomorphic to ω^{can} then we must have that $\mathrm{d}\omega = 0$. The converse is Darboux's theorem. A non-degenerate two form with $\mathrm{d}\omega = 0$ is called a *symplectic* form. An almost-Hermitian manifold with $\mathrm{d}\omega = 0$ is called an *almost-Kähler* manifold.

Finally, if J is locally isomorphic to J^{can} then the *Nijenhuis tensor*, defined as $N : TM \otimes TM \longrightarrow TM$ by

$$N(X,Y) = [X,Y] - J[X,JY] - J[JX,JY] + [JX,JY],$$

must be zero. The converse is given by the Newlander–Nirenberg theorem. An almost-complex structure with $N = 0$ is called a *complex* structure. An almost-Hermitian manifold with $N = 0$ is called a *Hermitian* manifold.

An almost-Hermitian manifold which satisfies both $\mathrm{d}\omega = 0$ and $N = 0$ is said to be Kähler. Of course, on a Kähler manifold one cannot simultaneously find a local trivialisation of both J and ω unless g is flat since J and ω suffice to determine g.

As remarked earlier, an almost-Hermitian structure corresponds to a reduction of the structure group to $\mathrm{U}(n)$. Thus almost Hermitian structures correspond to sections of the $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{U}(n)$ bundle associated to the oriented frame bundle. Since

$$\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{U}(n), \quad \mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}), \quad \mathrm{GL}^+(2n,\mathbb{R})/\mathrm{Sp}(n)$$

are all homotopic, there is a 1-1-1 correspondence between homotopy classes of almost-complex structures, homotopy classes of non-degenerate two forms and homotopy classes of almost-Hermitian structures. Indeed since

$$\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{SO}(2n)$$

is contractible, homotopy classes of almost-Hermitian structures are in 1-1 correspondence with homotopy classes of almost-complex structures compatible with a given metric. Notice that this also means that any almost-complex structure admits a compatible metric, as does any non-degenerate two form.

Thus to answer the question of whether or not there exists an almost-Hermitian structure on a given oriented manifold, M, one picks a metric g on M and asks if the associated $\mathrm{SO}(2n)/\mathrm{U}(n)$ bundle has a section. Thus the question is one of homotopy theory.

For example, in four dimensions we can easily write down some necessary conditions for the existence of an almost-Hermitian structure on a compact oriented manifold. An almost-complex structure J on M gives TM the structure of a complex vector bundle over M, we shall write $T^{1,0}$ for TM thought of in this way. So we have Chern classes $c_1(M) := c_1(T^{1,0})$ and $c_2(M) := c_2(T^{1,0})$. Since the underlying real bundle of $T^{1,0}$ is TM, we must have

$$\begin{aligned} c_1 &\equiv w_2(M) \mod 2 \\ c_2 &= e(M) \\ c_1^2 &= 2e(M) + p_1(M) \end{aligned}$$

where $w_2(M)$ is the second Stiefel–Whitney class of M, $e(M)$ is the Euler class of M and $p_1(M)$ is the first Pontrjagin class of M. Thus if M admits an almost-Hermitian structure compatible with its orientation there must be some $c \in H^2(M,\mathbb{Z})$ with $c \equiv w_2 \mod 2$ and $c^2 = 2e + p_1$. In fact it turns out that this is the only obstruction to the existence of almost-Hermitian structures.

Theorem 1.1.1 *[Wu48] If M is a compact oriented four manifold and if one has a cohomology class $c \in H^2(M,\mathbb{Z})$ with $c \equiv w_2(M) \mod 2$ and $c^2 = 2e(M) + p_1$ then there exists an almost-complex structure on M with $c_1 = c$.*

The correspondence is not one to one. Given an almost-complex structure J we can define another almost-complex structure $i(J)$ by forcing $i(J)$ to equal J outside the neighbourhood of a point and letting J be given by the non-zero element of $[S^4, \mathrm{SO}(4)/\mathrm{U}(2)] = [S^4, S^2] = \pi_4(S^2) = \mathbb{Z}_2$ inside the neighbourhood. It is proved in [Don90] that $i(J)$ is not homotopic to J even though the characteristic classes defined by J and $i(J)$ are equal.

The question of the existence of complex structures and of symplectic structures (equivalently of Hermitian and almost-Kähler structures) is far more subtle. For example it has been proved using gauge theory that, in dimension 4, the existence of complex structures and of symplectic structures depends not only on the homotopy type of the manifold but also on its diffeomorphism type — see section 2.3.

1.1.2 Representations of $\mathrm{U}(n)$

If we have a manifold M with structure group G, we can associate to each representation $\rho : \mathrm{G} \longrightarrow \mathrm{Aut}(V)$ of G a vector bundle which we shall denote $\underline{V}$ (see [KN63]). For example on an almost-Hermitian manifold (structure group $\mathrm{U}(n)$) the bundle $T^{1,0}$ arises from the standard action of $\mathrm{U}(n)$ on $\mathbb{C}^n$. Likewise, to any equivariant map between two representations, we can associate a bundle map between the associated bundles. Many of the natural maps that arise in almost-Hermitian geometry arise from equivariant maps — for example the map of TM to itself given by multiplication by J. With this and Schur's lemma in mind, we shall say:

Definition 1.1.2 *Two tensors $u \in U$ and $v \in V$ on a manifold with structure group* G *are* essentially equal *if there is a* G *equivariant map from U to*

V sending u to v and vice versa.

We shall see throughout the thesis that considering the decompositions of tensors into irreducibles provides a useful tool for calculations in almost-Hermitian geometry. Thus we give a brief summary of the aspects of the representation theory of $\mathrm{U}(n)$ that we shall use most.

Let M be a manifold with an almost-complex structure J and let $x \in M$.

Let $T_x^{1,0}$ be the complex vector space induced by J acting on T_xM. More precisely let $T_x^{1,0}$ be the $+i$ eigenspace of J acting on $T_xM \otimes_{\mathbb{R}} \mathbb{C}$ and let $T_x^{0,1}$ be the $-i$ eigenspace. Then $T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigwedge^{1,0} \oplus \bigwedge^{0,1}$ where $\bigwedge^{1,0}$ is the annihilator of $T^{0,1}$ and $\bigwedge^{0,1}$ is the annihilator of $T^{1,0}$ (we shall usually drop the point x from expressions such as T_xM).

We define $\bigwedge^{p,q}$ to be the subspace of $\bigwedge^{p+q}$ given by $\bigwedge^p \left(\bigwedge^{1,0}\right) \otimes \bigwedge^q \left(\bigwedge^{0,1}\right)$. Since $\bigwedge^{p,q}$ is conjugate to $\bigwedge^{q,p}$, $\bigwedge^{p,q} \oplus \bigwedge^{q,p}$ and $\bigwedge^{p,p}$ are both complexifications of real vector spaces which we shall call $[\![\bigwedge^{p,q}]\!]$ and $[\bigwedge^{p,p}]$ respectively. In general we shall follow [Sal89] by denoting the underlying real vector space of a complex vector V by $[V]$ and if V is the complexification of some real vector space W, we shall write $[\![V]\!] := W$. So, for example, $[\![\mathbb{C}]\!] = \mathbb{R} \oplus \mathbb{R}$ and $[\mathbb{C}] = \mathbb{R}$ (we are taking $\mathbb{R}$ and $\mathbb{C}$ to have the trivial group action).

We have the following isomorphisms of vector spaces:

$$\bigwedge^{2k} = \bigoplus_{p=0}^{k-1} [\![\bigwedge^{2k-p,p}]\!] \;\oplus\; [\bigwedge^{k,k}],$$

$$\bigwedge^{2k+1} = \bigoplus_{p=0}^{k} [\![\bigwedge^{2k+1-p,p}]\!].$$

Each of the vector spaces on the right hand side is irreducible under the action of $\mathrm{GL}(n, \mathbb{C})$, the structure group of an almost-complex manifold.

Of course, on an almost-Hermitian manifold, the structure group is $\mathrm{U}(n)$ and we wish to see how these vector spaces decompose further under the action of $\mathrm{U}(n)$.

The fundamental two form ω lies in $\bigwedge^{1,1}$, so we see that $\bigwedge^{1,1}$ splits equivariantly as $\bigwedge_0^{1,1} \oplus \mathbb{C}$ where $\bigwedge_0^{1,1}$ is the orthogonal complement of the span of ω and the $\mathbb{C}$ is spanned by ω. This is in fact a decomposition of $\bigwedge^{1,1}$

into irreducibles under $\mathrm{U}(n)$. In general we can define $\bigwedge_0^{p,q}$ to be the orthogonal complement of the image of $\bigwedge^{p-1,q-1}$ under wedging with ω. If $p+q \leq n$, then $\bigwedge_0^{p,q} \neq \{0\}$ and is in fact irreducible under $\mathrm{U}(n)$. Furthermore, if $\bigwedge_0^{p,q} \cong \bigwedge_0^{p',q'}$ then we must have $p = p'$ and $q = q'$ (unless $p+q > n$, $p'+q' > n$ in which case $\bigwedge_0^{p,q} = \bigwedge_0^{p',q'} = 0$). Note that $\bigwedge_0^{p,0} = \bigwedge^{p,0}$.

So we have the decomposition into irreducibles under $\mathrm{U}(n)$:

$$\bigwedge^{p,q} = \bigoplus_{r=0}^{min\{p,q\}} \bigwedge_0^{p-r,q-r}, \qquad p+q \leq n.$$

In Weyl's correspondence ([Ada69]), the space $\bigwedge_0^{p,q}$ is associated to the dominant weight:

$$(\underbrace{1,\ldots,1}_{p},0,\ldots,0,\underbrace{1,\ldots,1}_{q})$$

(with respect to the standard coordinates for the Lie algebra of the standard maximal torus of $\mathrm{U}(n)$).

Since it is known that the representation ring of $\mathrm{U}(n)$ is generated by the exterior powers of the standard representation of $\mathrm{U}(n)$, we can express any representation of $\mathrm{U}(n)$ as a sum of tensor products of these $\bigwedge_0^{p,q}$. Using the algorithm described in [Sal89] to decompose tensor products into irreducibles, we can then decompose these spaces into irreducibles under $\mathrm{U}(n)$. Indeed there exist computer programs to do this.

In principle then, we can decompose any $\mathrm{U}(n)$ representation into irreducibles. Thus we shall for the most part simply quote any results we need on $\mathrm{U}(n)$ decompositions of tensors. We should also remark that in the four dimensional case, spinors provide a simple route to the representation theory of both $\mathrm{SO}(4)$ and $\mathrm{U}(2)$ as we shall see in Chapter 2.

1.1.3 The torsion of an almost-Hermitian manifold

Let (M^{2n}, g, J) be an almost-Hermitian Manifold and let ∇ be its Levi-Civita connection. ∇ will not normally be unitary. Thus it is convenient to introduce a new covariant derivative $\overline{\nabla}$ given by:

$$\overline{\nabla}_X Y = \frac{1}{2}(\nabla_X Y - J\nabla_X JY)$$

which will always be unitary. The torsion of $\overline{\nabla}$ can be identified with the tensor:

$$\xi_X = \overline{\nabla}_X - \nabla_X = -\frac{1}{2}J(\nabla_X J).$$

ξ measures the extent to which ∇ fails to be unitary and is essentially equal to both ∇J and $\nabla\omega$. Differentiating the compatibility condition for a metric and an almost-complex structure tell us that $\nabla J \in T^*M \otimes \mathfrak{u}(n)^\perp \cong \bigwedge^1 \otimes [\![\bigwedge^{2,0}]\!]$ — where $\mathfrak{u}(n)^\perp$ is the orthogonal complement of $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$.

Lemma 1.1.3 *If $n \geq 3$, $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$ decomposes into irreducibles under* $\mathrm{U}(n)$ *as $\bigwedge^{3,0} \oplus A$. A is the irreducible* $\mathrm{U}(n)$ *module with dominant weight*

$$(2, 1, 0, \ldots, 0).$$

Thus we have

$$\begin{aligned}
\xi \in T^*M \otimes \mathfrak{u}(n)^\perp &\cong [\![\textstyle\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!] \\
&\cong [\![\textstyle\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!] \oplus [\![\bigwedge^{0,1} \otimes \bigwedge^{2,0}]\!] \\
&\cong [\![\textstyle\bigwedge^{3,0}]\!] \oplus [\![A]\!] \oplus [\![\bigwedge_0^{2,1}]\!] \oplus [\![\bigwedge^{1,0}]\!].
\end{aligned}$$

We write $\mathcal{W}_1 = [\![\bigwedge^{3,0}]\!]$, $\mathcal{W}_2 = [\![A]\!]$, $\mathcal{W}_3 = [\![\bigwedge_0^{2,1}]\!]$ and $\mathcal{W}_4 = [\![\bigwedge^{1,0}]\!]$. We have

Theorem 1.1.4 *[GH80] The tensor ξ belongs to the space $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, which reduces to $\mathcal{W}_2 \oplus \mathcal{W}_4$ when $n = 2$. Each $\mathcal{W}_i$ is* $\mathrm{U}(n)$ *irreducible. The spaces $\mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{W}_3 \oplus \mathcal{W}_4$ are* $\mathrm{GL}(n, \mathbb{C})$ *irreducible.*

Since $\mathrm{d}\omega \in \bigwedge^3 \cong [\![\bigwedge^{3,0}]\!] \oplus [\![\bigwedge_0^{1,1}]\!] \oplus [\![\bigwedge^{1,0}]\!]$ is given by an equivariant map applied to $\nabla\omega$ (as ∇ is torsion free), we must have that $\mathrm{d}\omega$ is essentially equal to the components of ξ in $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Similarly, up to an equivariant map, the Nijenhuis tensor N is determined by ∇J. An examination of the symmetries of the Nijenhuis tensor tells us that N lies in a space isomorphic to $[\![\mathrm{Hom}(\bigwedge^{0,1}, \bigwedge^{0,2})]\!] \cong [\![\bigwedge^{3,0}]\!] \oplus [\![A]\!]$. Hence we have

Proposition 1.1.5 *M is Hermitian if and only if $\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$, M is almost-Kähler if and only if $\xi \in \mathcal{W}_2$ and M is Kähler if and only if $\xi = 0$.*

Since in dimensions $n \geq 3$ there are four different components of ξ, one can define 16 different classes of almost-Hermitian manifold according to the vanishing of different components of ξ, many of which have been studied and given special names [GH80], [FFS94]. In dimension 4, the only special types of torsion tensor correspond to almost-Kähler, Hermitian and Kähler — all of which have the added interest of arising naturally as integrability conditions.

1.1.4 Geometry of almost-complex manifolds

The usual questions in almost-Hermitian geometry involve seeking interesting metrics compatible with either an integrable almost-complex structure or with an integrable fundamental 2-form. In this subsection we shall consider briefly the question of when a given almost-complex structure admits a compatible almost-Kähler structure. Thus we shall be considering the geometry of non-integrable almost-complex structures.

The basic invariant of an almost-complex structure is, of course, the Nijenhuis tensor. To be more precise, if one has two almost-complex structures J and J' defined around a point x and one seeks a map ϕ sending J to J' that keeps x fixed then one must at least be able to find a $\mathrm{GL}(n, \mathbb{C})$ equivariant map sending the Nijenhuis tensor of J at x to the Nijenhuis tensor of J' at x. Thus the Nijenhuis tensor provides a first-order obstruction to the existence of such a map.

If one examines the parallel question of finding a local isomorphism between two metrics, then there is no first-order obstruction to the existence of an isomorphism. The curvature, of course, provides a second-order obstruction. In the case of metrics it is clear that the covariant derivative of the curvature provides us with higher order obstructions.

There are also higher order obstructions to the existence of isomorphisms of almost-complex structures. However, because we lack a natural covariant derivative on an almost-complex manifold, these higher obstructions cannot be expressed in the convenient form of a tensor — instead one has to use the language of jet bundles. This perhaps gives one reason why the geometry of non-integrable almost-complex structures has not been much studied.

At this point it is worth remarking that we can also consider the Nijenhuis tensor as the map from $\bigwedge^{1,0} \longrightarrow \bigwedge^{0,2}$ given by taking the exterior derivative of a $(1,0)$ form and then projecting to $\bigwedge^{0,2}$. It is easy to check that this map is tensorial and then it follows immediately from Schur's lemma that this map must be essentially equal to the Nijenhuis tensor $N \in \bigwedge^{1,0} \otimes \bigwedge^{2,0}$. This definition of the Nijenhuis tensor is often easier to work with than the definition in terms of Lie brackets.

The result we wish to prove in this section is:

Theorem 1.1.6 *Not all complex structures in real dimensions greater than or equal to* 12 *admit a compatible almost-Kähler structure.*

Proof: Any algebraic tensor $T \in [\![\mathrm{Hom}(\bigwedge^{0,1}, \bigwedge^{2,0})]\!]$ *can be realised as the Nijenhuis tensor at some point of some almost-complex manifold. One can see this by writing the general complex structure up to first order as a power series about a point and then computing the Nijenhuis at that point*[1] .

Given a $T \in [\![\mathrm{Hom}(\bigwedge^{0,1}, \bigwedge^{2,0})]\!]$, *we can define a map*

$$\phi_T : [\bigwedge^{1,1}] \longrightarrow [\![\bigwedge^{3,0}]\!] \cong \mathcal{W}_1$$

by

$$\phi_T(\omega^{ij}) = T_\alpha^{[ij}\omega^{k]\alpha}.$$

This map is chosen such that if ω *is a non-degenerate two-form compatible with an almost-complex structure* J *(which implies* $\omega \in [\bigwedge^{1,1}]$*) and which has Nijenhuis tensor* T *at some point* x*, then the component of* ξ *in* $\mathcal{W}_1$ *at* x *is given by* $\phi_T(\omega)$*. The fact that* ϕ_T *has this property follows immediately from the proposition above. So if* (J, w) *is almost-Kähler, we must have* $\phi_T(\omega) = 0$.

However, $\dim[\bigwedge^{1,1}] = n^2$ *and* $\dim[\![\bigwedge^{3,0}]\!] = \frac{n(n-1)(n-2)}{3}$*. So if* $n \geq 6$*, we have* $\dim[\![\bigwedge^{3,0}]\!] \geq \dim[\bigwedge^{1,1}]$*. So if* $n \geq 6$ *we anticipate that for generic* T*,* ϕ_T *will be injective. In this case the condition* $\phi_T(\omega) = 0$ *would imply that* $\omega = 0$*, contradicting the non-degeneracy of* ω*.*

[1] More abstractly, using the ideas of Chapter 4, if we let E be the space of $A \in \mathrm{End}(TM)$ with $JA + AJ = 0$, then the symbol of the Nijenhuis tensor is a map:

$$T^*M \otimes E @ > \sigma(N) >> [\![\mathrm{Hom}(\bigwedge^{0,1}, \bigwedge^{2,0})]\!]$$

which must be onto as $[\![\mathrm{Hom}(\bigwedge^{0,1}, \bigwedge^{2,0})]\!]$ is $\mathrm{GL}(n, \mathbb{C})$ irreducible. The result follows immediately from this and the definition of the symbol.

All we require then is an example of a tensor T for which ϕ_T is injective. It is a simple matter to find an explicit example. The following Maple *program generates an example of such a T and checks that ϕ_T is injective.*

```
restart:
with(linalg):
n:=6:
```

This is the complex dimension we are working in. We begin by choosing an algebraic tensor $T \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ at random. We take

$$T = T[i,j,k]\frac{\partial}{\partial z^i} \otimes \mathrm{d}\overline{z}^j \wedge \mathrm{d}\overline{z}^k + \textit{conjugate}.$$

```
T:=array(sparse,1..n,1..n,1..n):
for a from 1 to n do:
for b from 1 to n do: for c from b+1 to n do:
T[a,b,c]:=rand(1..10)()+I*rand(1..10)():
T[a,c,b]:=-T[a,b,c]:
od:od:od:
```

We want to compute the map ϕ_T from this value of T. Of course ϕ_T is most conveniently thought of as a tensor lying in

$$[\bigwedge^{1,1}] \otimes [\![\bigwedge^{3,0}]\!]^* \subseteq T \otimes T \otimes T \otimes T \otimes T$$

and hence as a tensor with five indices. We wish however, to think of ϕ_T as a single matrix if we are to compute its rank. We first compute a complex n^2 by n^3 matrix called `phi` *which represents ϕ_T thought of as a map from $\bigwedge^{1,1}$ to $T^{1,0} \otimes T^{1,0} \otimes T^{1,0} \supseteq \bigwedge^{3,0}$. We do this by choosing obvious bases for $\bigwedge^{1,1}$ and $T^{1,0} \otimes T^{1,0} \otimes T^{1,0}$. With these preliminaries, it is easy to see that the following code computes* `phi`.

```
phi:=array(sparse,1..n^2,1..n^3):
for a from 1 to n do: for b from 1 to n do:
for c from 1 to n do: for d from 1 to n do:
phi[(a-1)*n + b , (d-1)*n^2 + (c-1)*n + b]:= phi[(a-1)*n + b ,
(d-1)*n^2 + (c-1)*n + b] + T[a,c,d]: phi[(a-1)*n + b ,
(d-1)*n^2 + (b-1)*n + c]:=
phi[(a-1)*n + b , (d-1)*n^2 + (b-1)*n + c] + T[a,d,c]:
phi[(a-1)*n + b , (b-1)*n^2 + (d-1)*n + c]:= phi[(a-1)*n + b ,
(b-1)*n^2 + (d-1)*n + c] + T[a,c,d]: od:od: od:od:
```

We are not quite done yet. We are only interested in the restriction of this map to $[\bigwedge^{1,1}]$. We compute a new matrix `psi` *which describes this.*

```
psi:=array(sparse,1..n^2,1..2*n^3): for a from 1 to
n^2 do: for b from 1 to n^3 do: psi[a,2*b]:=Re(phi[a,b]):
psi[a,2*b-1]:=Im(phi[a,b]): od:od:
```

(Although `psi` *represents a restriction of* `phi`*, it is in fact a larger matrix. The reason for this is that* `phi` *is a complex matrix and* `psi` *is a real matrix). Thus* `psi` *is a real matrix representing ϕ_T. In particular the rank of* `psi` *is equal to the rank of ϕ_T.*

```
rank(psi);
```

$$36$$

Which is equal to $n^2 = 36$ as we expected. Thus ϕ_T is injective.

Of course this program isn't guaranteed *to work! If one adds, straight after the "*`restart:`*", the line*

```
_seed:=040274:
```

then the same "random" numbers will be generated each time the program is run. In this case it is guaranteed to work. □

In dimensions 2 and 4 it is easy to see that any almost-complex structure locally admits a compatible almost-Kähler structure. The situation in dimensions 6, 8 and 10 is much more intricate. However, one conjectures that in these dimensions the generic almost-complex structure does not admit any compatible almost-Kähler structure.

We shall later consider the parallel questions of when a metric admits an almost-Kähler form and when a metric admits a Hermitian complex structure. The techniques used to consider the first of these questions could, in principle, be applied to find out whether or not the above conjecture is true. The essential complication is that one will be forced to look beyond just the Nijenhuis tensor and look at the higher order invariants of an almost-complex structure. As the reader will see in Chapter 3, looking at higher order obstructions can get rather involved. One has the added complication that these higher order invariants do not take the form of tensors — we shall get a flavour in Chapter 4 of how this would affect any exposition.

The author has made extensive use of Maple in exploring almost-Kähler geometry. The above example is typical in that the computer program is rather simple but rather lengthy. For this reason all Maple calculations will from now be restricted to Appendix A.

1.2 Curvature

1.2.1 Decomposition of the curvature tensor

Let (M^4, g) be an oriented, 4-dimensional Riemannian manifold and let R be its Riemann curvature tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The symmetries $R_{ijkl} = R_{jikl}$, $R_{ijkl} = R_{klij}$ of the curvature tensor tell us that we can view R as a self adjoint endomorphism of $\bigwedge^2$. Now recall that in 4 dimensions, the Hodge $* : \bigwedge^2 \longrightarrow \bigwedge^2$ defines a decomposition of $\bigwedge^2 \cong \bigwedge^+ \oplus \bigwedge^-$ into $+1$ and -1 eigenspaces of $*$ called the self-dual and anti-self-dual components of $\bigwedge^2$. This decomposition corresponds to the Lie algebra decomposition: $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Thus we can write the curvature tensor in block diagonal form as:

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12}\mathbf{1} & R_0 \\ \hline R_0^* & W^- + \frac{s'}{12}\mathbf{1} \end{array} \right),$$

where W^+ and W^- are self-adjoint and trace free. The only symmetry we have not yet fully exploited of the curvature tensor is the first Bianchi identity $R_{[ijk]l} = 0$. This tells us that $s = s'$. W^+ and W^- are called the self-dual and anti-self-dual parts of the Weyl tensor respectively. s is the scalar curvature of the manifold and R_0 is essentially equal to the trace-free part of the Ricci tensor. In fact this gives us a decomposition of the curvature into irreducibles under SO(4) [ST69] [AHS78]. This decomposition of the curvature is special to dimension 4. In other dimensions, the curvature has only three components: the scalar curvature, the trace free Ricci tensor and the Weyl tensor. We shall return to the topic of the decomposition of the Weyl tensor in Section 2.1.

If (M^4, g, J) is an almost-Hermitian manifold, we have the U(2) decompositions of $\bigwedge^+$ as $\mathbb{R} \oplus [\![\bigwedge^{2,0}]\!]$, where the $\mathbb{R}$ is spanned by ω, and $\bigwedge^-$ as $[\bigwedge_0^{1,1}]$. This allows us to refine the block decomposition as follows:

Definition 1.2.1 *We name various* U(2) *components of* R *as follows:*

$$R = \underbrace{\left(\begin{array}{c|c||c} a & W_F^+ & R_F \\ \hline (W_F^+)^* & W_{00}^+ + \frac{b}{2}\mathbf{1} & R_{00} \\ \hline\hline R_F^* & R_{00}^* & W^- + \frac{a+b}{3}\mathbf{1} \end{array}\right)}_{\mathbb{R} \qquad [\![\Lambda^{2,0}]\!] \qquad [\Lambda_0^{1,1}]}.$$

The double lines indicate the SO(4) *decomposition of the curvature.* W_{00}^+ *is self adjoint and trace free.*

If we call the U(2) modules where these various components lie $\mathcal{W}_{00}^+$, $\mathcal{W}_F^-$, $\mathcal{R}_F$, $\mathcal{R}_{00}$ and $\mathcal{W}^-$ and if we denote the space of algebraic curvature tensors by $\mathcal{R}$, we have a decomposition

$$\mathcal{R} \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathcal{W}_F^+ \oplus \mathcal{W}_{00}^+ \oplus \mathcal{W}^- \oplus \mathcal{R}_F \oplus \mathcal{R}_{00}$$

which it turns out ([TV91], [FFS94]) is a decomposition into irreducibles of U(2)[2].

This notation is decidedly non-standard but the author finds it helpful. It is chosen so as to make plain the relationship between this splitting and the splitting under SO(4). The reader may find it helpful to know that the F stands for form — both W_F^+ and R_F lie in components of $\bigwedge^2$. The 00 stands for "trace and form free".

Some of the pieces of the curvature have been given special names by other authors. For example $a \oplus W_F^+ \oplus R_F$ is the $*$-*Ricci tensor*. Its name presumably derives from the fact that on a Kähler manifold it is essentially equal to the Ricci form (as we shall see in a moment). Correspondingly, $4a$ is often called the $*$-scalar curvature and denoted s^*. We shall say that a manifold is weakly $*$-Einstein if W_F^+ and R_F both vanish. If in addition a is constant then the manifold is called strongly $*$-Einstein. It is not the case that weakly $*$-Einstein implies strongly $*$-Einstein; we shall see an example later. The phrase "$*$-Einstein" is used in the literature to mean one or other of the above with no real consensus emerging, so we shall never use the term without qualification.

[2]The reader may find it helpful to know that the spaces we have found can be identified with those given in [FFS94] as follows: $\mathcal{C}_6 = \mathcal{W_F}^+$, $\mathcal{C}_{-2} = \mathcal{R}_\mathcal{F}$, $\mathcal{C}_5 = \mathcal{W}_{00}^+$, $\mathcal{C}_8 = \mathcal{R}_{00}$ and $\mathcal{C}_3 = \mathcal{W}^-$.

As a useful first example, if the manifold is Kähler, then we must have that $R \in \mathfrak{u}(n) \otimes \mathfrak{u}(n) \cong ([\mathbb{R} \oplus \bigwedge_0^{1,1}] \otimes [\mathbb{R} \oplus \bigwedge_0^{1,1}])$. Pictorially,

$$R = \left(\begin{array}{c|c|c} * & 0 & * \\ \hline 0 & 0 & 0 \\ \hline\hline * & 0 & * \end{array} \right).$$

We denote the space of such tensors by $\mathcal{K}$, the space of algebraic Kähler curvature tensors. We observe instantly that on a Kähler four manifold the Ricci tensor is determined entirely by the Ricci form $a \oplus R_F \in \bigwedge^{1,1}$. Also we notice that if (M, g) admits three orthogonal Kähler structures I, J, K then W^+, s and R_0 must all vanish. Conversely if $W^+ = s = R_0$ then the induced connection on $\bigwedge^+$ is flat and hence we can find parallel, orthogonal complex structures I, J and K. In this situation, the manifold is called *hyperkähler*.

1.2.2 A relation between torsion and curvature

More generally, we are able to obtain information about the components of R in $\mathcal{K}^\perp$ from information on the torsion. Consider the Ricci identity:

$$\begin{aligned} R\omega &= \alpha(\nabla\nabla\omega) \\ &= \alpha(\nabla\xi) \\ &= \rho(\overline{\nabla}\xi) + \rho'(\xi \odot \xi) \end{aligned}$$

(α represents antisymmetrisation, ρ and ρ' are appropriate U(2) equivariant maps). We see that the components of R in $\mathcal{K}^\perp$ are determined by $\xi \odot \xi$ and $\overline{\nabla}\xi$. Recalling that $\xi \in \mathcal{W}_2 \oplus \mathcal{W}_4$ on a 4 dimensional almost-Hermitian manifold, we can write $\xi = \xi_2 + \xi_4$ in the obvious way. The following can now be proved by applying Schur's lemma:

Theorem 1.2.2 *[FFS94] The tensors $\overline{\nabla}\xi_i$ and $\xi_i \odot \xi_j$ contribute to the components of R in $\mathcal{K}^\perp$ if and only if there is a tick in the corresponding box in the table below.*

	b	W^+_{00}	W^+_F	R_{00}
$\xi_2 \odot \xi_2$	✓			
$\xi_2 \odot \xi_4$		✓	✓	✓
$\xi_4 \odot \xi_4$				✓
$\overline{\nabla}\xi_2$		✓	✓	✓
$\overline{\nabla}\xi_4$	✓		✓	✓

The result actually given in [FFS94] is a full analysis of the decomposition of the above Ricci identity, but in any dimension.

This result is of fundamental importance in almost-Kähler geometry and as such we shall meet it at a number of points in the thesis. For example in Section 3.3.1 we shall reprove the result in a more explicit form in the almost-Kähler case using Spinor notation. Also in Section 3.5, we shall prove the analogous result in the case of higher dimensional almost-Kähler manifolds.

We can extract a lot of very useful information directly from the above table.

Corollary 1.2.3 *A Hermitian 4-manifold has* $W^+_{00} = 0$ *(since then* $\xi_2 = 0$*).*

This gives a simple test of whether or not a Riemannian 4-manifold locally admits a compatible Hermitian structure:

The orbits of possible W^+ under SO(4) fall into three main classes:

1. $W^+ = 0$
2. W^+ has exactly two eigenvalues. In this case W^+ is called algebraically special and there is exactly one choice (up to sign) for J such that $W^+_{00} = 0$.
3. W^+ has distinct eigenvalues. It is easy to check that in this case there are exactly two (up to sign) choices of J which ensure that $W^+_{00} = 0$.

Suppose we are given a Riemannian manifold. If one has $W^+ \equiv 0$, it is a corollary of twistor theory that there exist many complex structures compatible with the metric — as we shall see later. At points where $W^+ \neq 0$ there are only a finite number of possibilities for J compatible with the metric for which W^+_{00}. One can simply check to see if these are integrable.

A question naturally raised by this test is whether or not it is possible for a four-dimensional Riemannian manifold to admit exactly two compatible complex structures. It turns out that it is — see [Kob95] and [Apo97].

In the case of almost-Kähler manifolds, the above table does not tell us anything quite so immediately useful. The most important observation is that on an almost-Kähler 4-manifold, b is completely determined by $\xi \odot \xi$. By Schur's lemma we see that for some universal constant, c, we must have $b = c\|\xi\|^2$. A calculation for any particular case gives that $c = 1$. In particular b must be non-positive on any almost-Kähler 4-manifold and strictly negative somewhere unless the manifold is Kähler.

Corollary 1.2.4 *An anti-self-dual metric (i.e. $W^+ = 0$) cannot admit a strictly almost-Kähler structure compatible with the orientation if the scalar curvature is positive (this includes constant curvature manifolds with positive scalar curvature).*

If an anti-self-dual metric with zero scalar curvature admits an almost-Kähler structure compatible with the orientation, it must be Kähler (this includes flat space).

Indeed any metric with a curvature tensor with suitably small W^+ proportionately large positive scalar curvature cannot admit a strictly almost-Kähler structure.

Any almost-Kähler 4-manifold with $R \in \mathcal{K}$ must be Kähler.

However, we can say relatively little about the negative scalar curvature case — and it will prove difficult to show even that hyperbolic space does not admit an almost-Kähler structure compatible with its metric. We shall examine this question in Chapter 3.

1.2.3 The geometry of the Nijenhuis tensor

In this section we give some simple applications of our relation between the various components of $\overline{\nabla}\xi$, $\xi \odot \xi$ and R to the geometry of the Nijenhuis tensor. The results obtained in this section are decidedly preliminary and will be improved upon greatly in Chapter 3. However, it is worth considering the geometry of the Nijenhuis tensor now as it will become surprisingly important.

Suppose we have an almost-Kähler four manifold whose Nijenhuis tensor at a given point does not vanish. We shall see that this effectively reduces the structure group to S^1. ξ, as we have commented before is, in this situation, effectively equal to the Nijenhuis tensor and lies in $[\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$. Let us write $\mathcal{D}$ for the kernel of ξ and $\mathcal{D}^\perp$ for its orthogonal complement.

Now let us pick any unit vector e^3 in $\mathcal{D}^\perp$ — note that there is an S^1 of possible choices for e^3. We define $e^4 = Je^3$. Then $0 \neq \xi_{e^3} = \Phi \in [\![\bigwedge^{2,0}]\!]$. Using the metric and rescaling, we can view Φ as an almost-complex structure which is orthogonal to J. This allows us to define $e^1 = \Phi e^3$ and finally $e^2 = Je^1$. Thus we have a reduction of the structure group to S^1 as claimed.[3]

Associated to this reduction of the structure group, we obtain additional geometric structures. We have already mentioned the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$. Another interesting geometric structure is a reverse oriented almost-Hermitian structure $\mathbb{J}$. This is obtained from the standard almost-complex structure J by reversing its sign on $\mathcal{D}^\perp$ but leaving the sign fixed on $\mathcal{D}$. Thus if we write the fundamental two form of $\mathbb{J}$ with respect to any of the bases described above we shall have $\omega^{\mathbb{J}} = e^1 \wedge e^2 - e^3 + e^4$.

These distributions are considered in [SV90] where they are shown to be involutive whenever M is almost-Kähler, Einstein, $*$-Einstein and has constant $*$-scalar curvature. In the first part of this section we shall weaken these conditions slightly. We shall then examine the zero set of the Nijenhuis tensor. We shall not prove anything about the almost-complex structure $\mathbb{J}$ in this section, but it should be noted that it will become important later.

Suppose that we have chosen a basis $\{e_1, e_2, e_3, e_4\}$ as above. Define $\Phi^1 = e^1 \wedge e^2 + e^3 \wedge e^4$, $\Psi^1 = e^1 \wedge e^2 - e^3 \wedge e^4$. Then extend Φ^1, Ψ^1 to orthonormal bases $\{\Phi^1, \Phi^2, \Phi^3\}$, $\{\Psi^1, \Psi^2, \Psi^3\}$ for $\bigwedge^+$, $\bigwedge^-$. In this notation we have:

Lemma 1.2.5 *If we define A and B as indicated below*

$$R = \underbrace{\left(\begin{array}{c|cc||c|cc} * & * & * & * & * & * \\ \hline \multirow{2}{*}{A} & * & * & \multirow{2}{*}{B} & * & * \\ & * & * & & * & * \\ \hline\hline * & * & * & * & * & * \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right)}_{\Phi^1\ \Phi^2\ \Phi^3\quad \Psi^1\ \Psi^2\ \Psi^3}$$

Then $\mathcal{D}$ is involutive iff $A + B = 0$. For example if R_{00} and W_F^+ vanish then $\mathcal{D}$ is involutive.

[3] This fact will resurface later when we introduce Spinor notation. The S^1 structure group is what we later refer to as a "choice of gauge".

Proof: *Suppose that X, $Y \in \mathcal{D}$. Then*

$$\begin{aligned}\xi_{[X,Y]} &= \xi_{\nabla_X Y - \nabla_Y X} \\ &= \nabla_X(\xi_Y) - \nabla_Y(\xi_X) - (\nabla_X \xi)_Y + (\nabla_Y \xi)_X \\ &= -(\nabla_X \xi)_Y + (\nabla_Y \xi)_X \\ &= -(\overline{\nabla}_X \xi)_Y + (\overline{\nabla}_Y \xi)_X\end{aligned}$$

since $\xi_X = \xi_Y = 0$ everywhere. So $\mathcal{D}$ is involutive iff $\alpha(\overline{\nabla}\xi)(e_1, e_2) = 0$. But $\alpha(\overline{\nabla}\xi)(e_1, e_2) = A + B$ by Theorem 1.2.2. □

Thus if (M^4, g, J) is almost-Kähler and Einstein then $\mathcal{D}$ is involutive if and only if M^4 is weakly $*$-Einstein.

This result is as good as we are going to get for the time being. However, it is at least suggestive of the importance of the distributions on weakly $*$-Einstein manifolds.

We shall now examine the zero set of the Nijenhuis tensor. We shall say that the zero set is *J-invariant* if $\nabla_X \xi = 0$ implies $\nabla_{JX} \xi = 0$ at each point where ξ is zero. If the zero set is also a sub-manifold (for example if ξ is transverse to zero) then the zero set becomes a complex manifold with the induced complex structure. On an almost-Kähler manifold the zero set would then be Kähler with the induced metric — in fact this is true on an almost-Hermitian manifold with $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_2$ — so called $(1,2)$-symplectic manifolds.

On an almost-Kähler 4-manifold, by dimension counting, one expects that the zero set will be 0-dimensional. Thus we shall consider higher dimensional manifolds. Our block decomposition of the curvature tensor is still valid in dimensions greater than four. However, the decomposition is no longer into irreducibles and the correspondence between the $\mathrm{SO}(2n)$ decomposition and the $\mathrm{U}(n)$ decomposition will be more complicated. For example the tensor R_{00} is no longer determined entirely by the Ricci tensor. Nevertheless the tensors W_F^+ and R_{00} are still well-defined. So we can state:

Proposition 1.2.6 *If (M, g, J) is a (1,2)-symplectic manifold with $R_{00} = 0$ and $W_F^+ = 0$ then the zero set of ξ is J-invariant.* [4]

[4] In the notation of [FFS94], the condition that $W_F^+ = R_{00} = 0$ is that the curvature have no components in $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$. This is a $\mathrm{GL}(n, \mathbb{C})$ submodule of R

Proof: *By definition of a (1,2)-symplectic manifold, $\xi \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$. We can write $\xi = \eta + \overline{\eta}$ with $\eta \in \bigwedge^{1,0} \otimes \bigwedge^{2,0}$. Then*

$$\begin{aligned}
\overline{\nabla}\eta &\in (\bigwedge^{1,0} \oplus \bigwedge^{0,1}) \otimes (\bigwedge^{1,0} \otimes \bigwedge^{2,0}) \\
&\cong (\bigwedge^{1,0} \otimes (\bigwedge^{1,0} \otimes \bigwedge^{2,0})) \oplus (\bigwedge^{0,1} \otimes (\bigwedge^{1,0} \otimes \bigwedge^{2,0})) \\
&\cong \underbrace{\bigwedge^{2,0} \otimes \bigwedge^{2,0}}_{\cong \mathcal{W}_{00}^{+}} \oplus \operatorname{Ker} \alpha \oplus \underbrace{\bigwedge_{0}^{1,1} \otimes \bigwedge^{2,0}}_{\cong \mathcal{R}_{00}} \oplus \underbrace{\mathbb{C} \otimes \bigwedge^{2,0}}_{\cong \mathcal{W}_{F}^{+}}
\end{aligned}$$

where α maps $T^*M \otimes T^*M \otimes \bigwedge^{2,0} \longrightarrow \bigwedge^2 \otimes \bigwedge^{2,0}$ by antisymmetrisation on the first two factors.

Define ψ mapping $T^*M \otimes T^*M \otimes \bigwedge^{2,0} \longrightarrow \bigwedge^2 \otimes \bigwedge^{2,0}$ by $\psi(A)(X,Y) = A(JX, JY)$ for all $A \in T^*M \otimes T^*M \otimes \bigwedge^{2,0}$, X, $Y \in TM$. By definition of $\bigwedge^{1,0}$, $\bigwedge^{0,1}$ we see that ψ acts on $\bigwedge^{0,1} \otimes \bigwedge^{1,0} \otimes \bigwedge^{2,0}$ as the identity and on $\bigwedge^{1,0} \otimes \bigwedge^{1,0} \otimes \bigwedge^{2,0}$ as minus the identity.

Hence if $R_{00} = 0$ and $W_F^+ = 0$ then $\psi(\overline{\nabla}\eta) = -(\overline{\nabla}\eta)$. So we have that $(\overline{\nabla}_{JX}\xi)_{JY} + (\overline{\nabla}_X\xi)_Y = 0$ for all X, $Y \in TM$. When $\xi = 0$, $\overline{\nabla} = \nabla$. It follows that at zeros of ξ, $\nabla_X \xi = 0$ implies $\nabla_{JX}\xi = 0$. $\square$

In the case where M is nearly Kähler (that is ξ lies entirely in $\mathcal{W}_1$) we can say even more. By Theorem 5.4 in [FFS94], the components of $\overline{\nabla}\xi$ in $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ vanish in the nearly Kähler case. Thus on a nearly Kähler manifold, the zero set is always J-invariant.

This reasoning can be pushed a little further to recover:

Theorem 1.2.7 *[Gra70] Let M be a nearly Kähler manifold. For each $m \in M$ let*

$$\mathcal{H}(m) = \{X \in T_mM | \nabla_X J = 0\}$$

then whenever $\mathcal{H}$ has constant dimension, it is an involutive distribution whose integral sub-manifolds are Kähler sub-manifolds of M.

1.3 Ansätze

1.3.1 The Gibbons–Hawking ansatz and Tod's construction

Suppose that (M^4, g, I, J, K) is a hyperkähler manifold with a Killing vector field X. X must induce an isometry on the two sphere in $\bigwedge^+$ with axes I, J, K. Either X will leave the sphere fixed — in which case we shall say that X is a *translational* Killing vector field — or else X will rotate the sphere about some axis in which case we shall call X a *rotational* Killing vector field.

Let us suppose first of all that X is a translational Killing vector field. Then because it preserves each of the Kähler forms associated to I, J and K we shall obtain three moment maps which we shall call x, y, z [HKLR87]. If we choose one further coordinate t such that $\frac{\partial}{\partial t} = X$ and if we write U for the length of X then we are able to write the metric as:

$$g = U(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{1}{U}(\mathrm{d}t + \theta)^2. \tag{1.1}$$

where θ is defined by insisting that $\mathrm{d}t + \theta$ is the form associated to X by the metric. Making a different choice for t changes θ by adding some exact form. We shall refer to this choice as a choice of gauge.

With this notation, we can write out the two-forms associated to I, J and K as:

$$\omega^I = \mathrm{d}x \wedge (\mathrm{d}t + \theta) + U\mathrm{d}y \wedge \mathrm{d}z \tag{1.2}$$

$$\omega^J = \mathrm{d}y \wedge (\mathrm{d}t + \theta) + U\mathrm{d}z \wedge \mathrm{d}x \tag{1.3}$$

$$\omega^K = \mathrm{d}z \wedge (\mathrm{d}t + \theta) + U\mathrm{d}x \wedge \mathrm{d}y \tag{1.4}$$

If we write

$$(\mathrm{d}\theta) = \theta_x \mathrm{d}y \wedge \mathrm{d}z + \theta_y \mathrm{d}z \wedge \mathrm{d}x + \theta_z \mathrm{d}x \wedge \mathrm{d}y$$

(the subscripts are chosen for ease of notation), then we can differentiate our formulae for the ω's to see that:

$$0 = \mathrm{d}\omega = (-\theta_x + U_x)(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$$

so $\theta_x = U_x$. Similarly, $\theta_y = U_y$ and $\theta_z = U_z$. So we have that

$$(\mathrm{d}\theta) = U_x \mathrm{d}y \wedge \mathrm{d}z + U_y \mathrm{d}z \wedge \mathrm{d}x + U_z \mathrm{d}x \wedge \mathrm{d}y. \tag{1.5}$$

Taking d of this we see that we must have $U_{xx} + U_{yy} + U_{zz} = 0$. Conversely if we have chosen a positive, harmonic, U defined on some region of $\mathbb{R}^3$ and choose a 1-form θ on $\mathbb{R}^4$ such that $(\mathrm{d}\theta)$ is given by (1.5) then the metric (1.1) will necessarily be hyperkähler since the forms (1.2 etc.) will all be closed.

This correspondence between hyperkähler manifolds and harmonic forms is known as the Gibbons–Hawking ansatz [GH78].

A special case of the Gibbons–Hawking ansatz provides the only known examples of strictly almost-Kähler, Einstein 4-manifolds:

There is a two sphere of "obvious" almost-complex structures compatible with the reverse orientation on any hyperkähler manifold with a Killing vector field. One simply picks a covariant constant almost-complex structure, without loss of generality I, and changes its sign on the two-plane spanned by X and IX. We shall call the new almost-complex structure I'. The 2-form associated to the reverse oriented almost-complex structure I' is given by:

$$\omega^{I'} = (\mathrm{d}t + \theta) \wedge \mathrm{d}x + U \mathrm{d}y \wedge \mathrm{d}z$$

One computes $\mathrm{d}\omega^{I'}$ to obtain:

Proposition 1.3.1 *[Tod97a] I' is almost-Kähler if and only if $U_x = 0$. Thus from a positive harmonic function of two variables $U(y, z)$ we can obtain a metric that is hyperkähler with one orientation (and hence Einstein) and which is almost-Kähler with respect to the other.*

We shall refer to this method of generating almost-Kähler, Einstein metrics as Tod's construction. This is the only known method of generating 4 dimensional strictly almost-Kähler, Einstein manifolds and, as such, we shall devote quite a lot of attention to it. Note that the first example found of a strictly almost-Kähler, Einstein manifold was given by Przanowski and Nurowski in [PN]. They observed that their metric was a special case of the Gibbons-Hawking ansatz, which is what motivated Paul Tod to devise his more general examples.

Tod's examples provide us with a good opportunity to see an explicit example of almost-Kähler geometry. The structure of Tod's examples is surprisingly rich. To examine these examples first note that U must be the real part of some holomorphic function $U + iV$ of y and z. It is easy to see that

$V\mathrm{d}x$ provides one possible choice for θ. If we expand U and V as power series around the origin, the Cauchy–Riemann relations tell us that we may write:

$$U = U_0 + U_y y + U_z z + \frac{1}{2}V_{yz}y^2 + U_{yz}yz - \frac{1}{2}V_{yz}z^2 + \dots$$
$$V = V_0 - U_z y + U_y z - \frac{1}{2}U_{yz}y^2 + V_{yz}yz + \frac{1}{2}U_{yz}z^2 + \dots$$

where V_0, U_0, U_y, U_z, U_{yz}, V_{yz} are real constants. Conversely any such two-jet functions can be extended to a holomorphic function of y and z. Of course, these first few orders are all that one needs to compute the curvature tensor. If one does this and then writes the curvature as a matrix with respect to the basis $\omega^{I'}$, $\omega^{J'}$, $\omega^{K'}$, ω^I, ω^J, ω^K for $\bigwedge^2$, one sees that

$$R = \left(\begin{array}{c|cc||ccc} 8\frac{U_z^2+U_y^2}{U_0^3} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 8\frac{U_z^2+U_0V_{yz}-2U_y^2}{U_0^3} & 8\frac{3U_yU_z-U_{yz}U_0}{U_0^3} & 0 & 0 & 0 \\ 0 & 8\frac{3U_yU_z-U_{yz}U_0}{U_0^3} & -8\frac{2U_z^2+U_0V_{yz}-U_y^2}{U_0^3} & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In particular, M is always weakly $*$-Einstein, and $\omega^{I'}$ is Kähler if and only if U and V are constant (since we know it is Ricci flat and self-dual, it could only be Kähler if it were flat in which case we would have $U_z^2 + U_y^2 \equiv 0$).

Of course, a more natural way to check if $\omega^{I'}$ is Kähler would be to compute the Nijenhuis tensor. We view the Nijenhuis tensor as the map $\bigwedge^{1,0} \longrightarrow \bigwedge^{0,2}$ given by first taking the exterior derivative and then projecting to $\bigwedge^{0,2}$. We note that $\bigwedge^{1,0}$ is spanned by $a = (\mathrm{d}t + V\mathrm{d}x) + iU\mathrm{d}x$ and $b = \mathrm{d}y + i\mathrm{d}z$. On the other hand, the dual space $(\bigwedge^{0,1})^* \subseteq TM$ is spanned by

$$X = \frac{\partial}{\partial t} + i\left(U\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right),$$

$$Y = \frac{\partial}{\partial y} + I\frac{\partial}{\partial z}.$$

Thus the complex structure $\omega^{I'}$ is integrable if and only if $\mathrm{d}a(X,Y) = 0$ and $\mathrm{d}b(X,Y) = 0$. Of course, $\mathrm{d}b = 0$ and it is easy to check that $\mathrm{d}a(X,Y) = 2U_y - 2iU_z$, confirming our result. Note that the kernel of the Nijenhuis tensor is spanned by b.

The Nijenhuis tensor is, of course, essentially equal to $\xi \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$. The kernel of ξ is the distribution $\mathcal{D}$ mentioned earlier. In the case of Tod's examples, one observes that $\mathcal{D}$ and $\mathcal{D}^\perp$ are both involutive distributions. Also the distribution $\mathcal{D}^\perp$ is spanned by the two Killing vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$.

All these observations about the geometry of Tod's examples may seem rather ad hoc. After all, we have not yet obtained any particularly significant categorisation of these examples. However, these observations will prove central to the proof of the following result:

Theorem 1.3.2 *All strictly almost-Kähler, Einstein manifolds which are in addition weakly $*$-Einstein are given by Tod's construction.*

This will be the main result of Chapter 3. As such, these simple remarks on Tod's examples have surprisingly strong implications.

1.3.2 Related constructions

We begin by describing a system of coordinates one can use on a Kähler manifold (M^4, g, J) with a Killing vector field which preserves J. These coordinates were first exploited by LeBrun in [LeB91b]. Firstly, we have a moment map z associated to the Killing vector field. Since J is a complex structure and is preserved by J, the distribution spanned by $\langle X, JX \rangle$ will be integrable and, furthermore, J will induce a complex structure on Σ, the space of leaves. This is of course just the standard complex quotient construction. If we pick holomorphic coordinates $x + iy$ for Σ then we now have three real coordinates x, y, z on M. We pick one further coordinate t by insisting that $\frac{\partial}{\partial t} = X$. We now introduce a function V by $V = \frac{1}{\|x\|}$, a form θ so that the form associated by the metric to X is $V(\mathrm{d}t + \theta)$ and one further real valued function u which allows us to evaluate the length of $\frac{\partial}{\partial x}$. One can then write:

$$g = V(e^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2) + \frac{1}{V}(\mathrm{d}t + \theta)^2.$$

Once again, a change of choice for the coordinate t changes θ by the addition of an exact form.

The Kähler form is given by:

$$\omega = \mathrm{d}z \wedge (\mathrm{d}t + \theta) + Ve^u(\mathrm{d}x \wedge \mathrm{d}y).$$

One can easily check from this that the conditions that ω describes an integrable complex structure together with the the condition $\mathrm{d}\omega = 0$ are equivalent to:

$$\mathrm{d}\theta = V_x \mathrm{d}y \wedge \mathrm{d}z + V_y \mathrm{d}z \wedge \mathrm{d}x + (Ve^u)_z \mathrm{d}x \wedge \mathrm{d}y$$

which determines θ completely up to our gauge freedom. Taking d of this equation we have the integrability condition:

$$V_{xx} + V_{yy} + (Ve^u)_{zz} = 0. \tag{1.6}$$

Thus equation (1.6) describes Kähler 4-manifolds with a Killing vector field. If one now imposes conditions on the Ricci tensor, one can readily calculate what the new conditions on u and V are:

Due to:		Conditions on u and V:
[LeB91b]	$s = 0$	$u_{xx} + u_{yy} + (e^u)_{zz} = 0$
[PP91]	$R_0 = 0$	$V = \frac{u_z}{sz+M}$, $u_{xx} + u_{yy} + (e^u)_{zz} = \frac{2s(e^u)_z}{sz+M}$
[BF82]	$R_0 = 0$, $s = 0$ and X is rotational	$V = cu_z$, $u_{xx} + u_{yy} + (e^u)_{zz} = 0$

where c and M are constants. In the last two cases (1.6) is automatic. The first case was exploited by [LeB91b] to find compact examples of scalar flat Kähler manifolds. Once one knows a solution to the so-called SU(∞) Toda field equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

one is only faced with the comparatively simple problem of solving the linear equation (1.6) for V. This allows one, for example, to superpose known solutions to obtain new metrics.

Another interesting application of this construction is to ASD, Einstein manifolds with a Killing vector field. The problem that we do not have any obvious Kähler structure is overcome by the observation that $(\nabla X)^+$ is conformal to a Kähler structure. One can now run through a similar analysis to see that:

$$g = \frac{V}{z^2}\left[e^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2\right] + \frac{1}{Vz^2}(\mathrm{d}t + \theta)^2$$

where $-2sV = 2 - zu_z$, $u_{xx} + u_{yy} + (e^u)_{zz} = 0$ and

$$\mathrm{d}\theta = -V_x \mathrm{d}y \wedge \mathrm{d}z - V_y \mathrm{d}z \wedge \mathrm{d}x - (Ve^u)_z DEx \wedge \mathrm{d}y.$$

This is shown in [Tod97b] and [Prz91].

The primary interest of these constructions in this thesis is that we shall obtain analogous formulae in Chapter 4 for strictly-Hermitian, Einstein 4-Manifolds. However, one immediate question is "can we generalise Tod's construction in this context?" The answer is no — at least not in any way obvious to the author. For example:

Lemma 1.3.3 *If (M^4, g, I, J, K) is hyperkähler with a* rotational *Killing vector field X preserving I, then the almost-complex structure I' defined from I via X is never strictly almost-Kähler.*

Proof: We use the above construction replacing J by I throughout. We have that

$$\omega^{I'} = (\mathrm{d}t + \theta) \wedge \mathrm{d}z + Ve^u(\mathrm{d}x \wedge \mathrm{d}y).$$

Differentiating we have:

$$\mathrm{d}\omega^{I'} = (2Ve^u)_z \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

So I' is almost-Kähler if and only if $(Ve^u)_z = 0$. But $V = cu_z$ so this is equivalent to insisting that $(cu_z e^u)_z = 0$ and hence that $(e^u)_{zz} = u_{xx} + u_{yy} = 0$. We can choose the holomorphic coordinates on Σ such that $e^u = Az + B$ for constants A and B. It follows readily that the manifold is flat and hence Kähler. $\square$

1.3.3 Riemannian submersions

One situation in which almost-Kähler, Einstein structures naturally arise is when one attempts to manufacture Einstein manifolds by considering Riemannian submersions. Indeed, the fact that almost-Kähler Einstein structures are important in this context provides an important piece of motivation for their study. For this reason, we shall devote this and the next section to a brief study of Riemannian submersions. The material is a synopsis of the parts of Chapter 9 of [Bes87] on Riemannian submersions relevant to almost-Kähler geometry.

Suppose we have a map $\pi : (M, g) \longrightarrow (B, \check{g})$ between two Riemannian manifolds. Given a point b in the *base*, B, we have the *fibre* $F_b = \pi^{-1}(b)$ above b. At a point x in the fibre we can use the metric to decompose $T_x M$ as $\mathcal{V}_x \oplus \mathcal{H}_x$, where $\mathcal{V}_x$ is the tangent space to the fibre and $\mathcal{H}_x$ is its orthogonal complement. $\mathcal{V}$ and $\mathcal{H}$ are called the vertical and horizontal spaces respectively. π is called a *Riemannian submersion* if π_* defines an isometry from $\mathcal{H}_x$ to $T_x B$ at every point x in M.

Let us write ∇ for the Levi–Civita connection of (M, g), $\hat{\nabla}$ for the Levi–Civita connections of $(F_b, \hat{g})$ with their induced metrics and $\check{D}$ for the Levi–Civita connection of the base. We shall also write $\mathcal{H}$ and $\mathcal{V}$ for the projections of M to the horizontal and vertical subspaces. With this notation we can define tensorial invariants A and T of a Riemannian submersion as follows:

$$T_X Y = \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y$$

T is clearly related to the second fundamental form of each fibre and hence vanishes if and only if each fibre is totally geodesic. We shall call such a fibration a *totally geodesic fibration*.

Whilst $\mathcal{V}$ is necessarily an integrable distribution, $\mathcal{H}$ need not be. We define the second tensorial invariant by:

$$A_X Y = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y.$$

It measures the failure of $\mathcal{H}$ to be integrable in that:

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$$

whenever X, Y are horizontal vector fields.

A theorem due to [Her60] and its converse due to [Vil70] tell us that totally geodesic submersions can be thought of as bundles $M \longrightarrow B$ with isometric fibres and structure group some subgroup of the isometry group of the fibre. If one then picks any connection $\mathcal{H}$ for M and a metric on B, then one induces a metric on M by insisting that $\mathcal{H}$ is isometric to TB and using metric on F to define the metric on $\mathcal{V}$. In this correspondence between totally geodesic Riemannian submersions and connections, one sees that the tensor A is the curvature of the connection.

One can calculate the curvature of the total space in terms of A, T and the curvature of the basis. The formulae are rather complex. Thus in studying Riemannian submersions one usually restricts to a specific class of Riemannian submersions. For example if we define $\check{\delta} A = -\sum_i (\nabla_{X_i} A) X_i$ with X_i an orthonormal basis for $\mathcal{H}_x$ one can prove:

Proposition 1.3.4 *If π is a Riemannian submersion with totally geodesic fibres, then (M, g) is Einstein if and only if, for some constant λ, the Ricci tensors $\check{r}$ and $\hat{r}$ of B and F satisfy*

$$\check{\delta} A = 0$$

in which case $\mathcal{H}$ is called a Yang–Mills connection

$$\hat{r}(U, V) + (AU, AV) = \lambda(U, V)$$

for vertical vectors U and V and

$$\check{r}(X, Y) - 2(A_X, A_Y) = \lambda(X, Y)$$

for horizontal vectors X and Y.

It is in the Yang–Mills condition that we see almost-Kähler structures begin to emerge. For example if M is a principal S^1 bundle over B then the curvature Ω of any principal connection is known to be the pull-back to M of a closed 2-form ω on B. It is easy to check that $\check{\delta} A = 0$ if and only if ω is co-closed.

We can easily adapt the above Proposition in this case to give:

Corollary 1.3.5 *If $\pi : M \longrightarrow B$ is a principal S^1 bundle with totally geodesic fibres of length 2π and if ω is the two form on B whose pull-back gives the curvature of the connection then (M, g) is Einstein if and only if:*

$$(\check{r}(X, Y)) - \frac{1}{2}(\omega_X, \omega_Y) = \frac{|\omega|^2}{4}(X, Y)$$

and ω is closed, co-closed and has constant norm.

Suppose we insist further that $(B, \check{g})$ is compact and Einstein. Then principal S^1 bundles over B are classified by the cohomology group $H^2(B, \mathbb{Z})$. Indeed if $\alpha \in H^2(B, \mathbb{Z})$ classifies a principal bundle with connection $\mathcal{H}$ and curvature ω, we have that $[\omega] = \mathbb{R}(2\pi\alpha)$ where $\mathbb{R} : H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$ is the change of coefficients morphism. Conversely any closed ω with the appropriate cohomology represents the curvature of some connection on the principal bundle. One easily obtains:

Theorem 1.3.6 *Let $(B, \check{g})$ be a compact Einstein manifold (with scalar curvature $\check{\lambda}$ and $\pi : M \longrightarrow B$ a principal S^1 bundle classified by α then M admits a unique S^1 invariant Einstein metric such that π is a totally geodesic submersion if and only if either:*

1. *$\check{\lambda} = 0$, $\alpha^{\mathbb{R}} = 0$ in which case a finite covering of M is the Riemannian product $B \times \mathrm{S}^1$,*
2. *$\check{\lambda} > 0$ and there exists on B an almost-Kähler structure $(J, \check{g}, \omega')$ such that $[\omega']$ is a multiple of $\mathbb{R}(\alpha)$.*

Of course, in the above theorem ω' is just an appropriate multiple of the constant norm ω.

In fact, as a construction of compact manifolds one might as well replace the almost-Kähler condition by simply Kähler since as we shall see any almost-Kähler, Einstein manifold with non-negative scalar curvature is necessarily Kähler. In the negative scalar curvature case one can still produce an Einstein manifold with a *Lorentzian* metric.

This theorem shows the relevance of almost-Kähler structures to the study of certain Riemannian submersions. A more interesting example is given in the next section.

1.3.4 Bérard Bergery's construction

The material in this section is based on [Ber82].

Let $(B^{n-2}, \check{g}, \omega)$ be an almost-Kähler, Einstein manifold with positive scalar curvature with metric normalised such that $\check{r} = n\check{g}$ and assume that there is an indivisible integral cohomology class α with $n[\omega] = 2\pi q \mathbb{R}(\alpha)$ for some positive q.

Let us denote by $P(s)$ the total space of the S^1 bundle over B classified by $s\alpha$ in $H^2(B, \mathbb{Z})$ and let $M(s)$ be the S^2 bundle associated to $P(s)$ via the action of S^1 on S^2 by rotation about the North-South axis.

We define metrics $g(a, b)$ on $P(s)$ such that $\pi : P(s) \longrightarrow B$ is a Riemannian submersion with fibres of length a and metric $b\check{g}$ on the base. If we view $M(s)$ as a quotient of the manifold with boundary $[0, l] \otimes P(s)$ we can consider metrics of the form $\mathrm{d}t^2 + g(f(t), h(t))$ where t is a coordinate on $[0, l]$, and f, h are real valued functions on $[0, l]$.

One can now compute the Ricci curvature of this metric to find that the condition that M with this metric is Einstein is determined by ODE's. Bérard Bergery goes on to solve these equations — indeed he manages to solve them with the appropriate boundary conditions to ensure that the metric can be passed to the quotient. The result is:

Theorem 1.3.7 *In the above situation, if $1 \leq s \leq q$ then $M(s)$ admits an Einstein metric of positive scalar curvature.*

This construction for example gives the Page metric on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which is the non-trivial sphere bundle over S^2. The almost complex structure on B induces a canonical almost-complex structure over M. This almost-complex structure will be complex if and only if the almost-Kähler structure on the base is Kähler.

Of course, as we mentioned earlier it has been proved that a compact, Einstein almost-Kähler manifold is necessarily Kähler. However, what is interesting to us is that the local calculation only requires the almost-Kähler condition. In this light, the complete case is perhaps of more interest. Let us denote by $E(s)$ the quotient of $[0, \infty) \times P(s)$ by the fibration $\{0\} \times P(s) \longrightarrow B$. Let us once again consider metrics of the form $\mathrm{d}t^2 + g(f(t), h(t))$. We quote Bérard Bergery's result:

Theorem 1.3.8 *Let $(B, \check{g}, J, \omega)$ be a compact, Einstein and almost-Kähler manifold with scalar curvature $\check{\lambda}$. The above construction yields a complete Einstein metric on $E(s)$ and $P(s) \times \mathbb{R}$ in precisely the following situations:*

1. *for any $\check{\lambda}$ and any $s \geq 1$, $E(s)$ admits a one-parameter family of (non-homothetic) complete Einstein almost-Hermitian metrics (with $\lambda < 0$);*
2. *if $\check{\lambda} \leq 0$, $s \geq 1$ then $P(s) \times \mathbb{R}$ admits a one-parameter family of complete Einstein almost-Hermitian metrics (with $\lambda < 0$);*
3. *if $\check{\lambda} > 0$, $E(s)$ also admits:*
    - *for $s < q$, a complete Ricci-flat almost-Hermitian metric;*

- *for $s = q$, a complete Ricci-flat almost-Kähler metric;*
- *for $s > q$, a complete Einstein almost-Kähler metric (with $\lambda < 0$);*

4. *if $\check{\lambda} \leq 0$, $E(s)$ and $P(s) \times \mathbb{R}$ admit furthermore a complete Einstein almost-Kähler metric (with $\lambda < 0$).*

Moreover, if B is Kähler, these almost-Hermitian metrics are Hermitian and the almost-Kähler metrics are Kähler.

Certainly this theorem provides a good deal of motivation for studying almost-Kähler geometry. However, we cannot make use of the fact that B need only be almost-Kähler since we do not know of any compact almost-Kähler manifolds which are not Kähler. However, one observation is worth making. If one is willing to drop the requirement of completeness, one can view Bérard Bergery's construction as entirely local and observe that one must be able to construct from Tod's examples (non-complete) 6-dimensional almost-Kähler, Einstein manifolds. This is the only known way to produce strictly almost-Kähler, Einstein manifolds with negative scalar curvature. Indeed one can continue in this way to produce local, non-product examples of strictly almost-Kähler, Einstein manifolds in any even dimension ≥ 4.

Chapter 2

Special features of the four dimensional case

2.1 Spinors and self-duality

There is a distinct prejudice in this thesis towards the consideration of four-dimensional manifolds. One of the reasons for this is practical rather than theoretical — in lower dimensions the algebra is much easier to handle purely because the size of tensors is smaller. However, there are a number of theoretical reasons for paying particularly close attention to the four-dimensional case, and we shall explore these in this chapter. Many of these stem from the twin concepts of spinors and self-duality.

As was mentioned earlier, the decomposition $\bigwedge^2 \cong \bigwedge^+ \oplus \bigwedge^-$ arises from the Lie algebra decomposition $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ which in turn arises from the fact that the double cover, Spin(4) of SO(4) is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This leads to the decomposition of the Weyl tensor W in four dimensions — a phenomenon which does not occur in any other dimension. There are interesting links between the concept of self-duality and almost-Hermitian geometry in 4-dimensions, some of which we shall explore in this section.

We should remark that the use of spinors in geometry was pioneered by Penrose [PR86a], [PR86b]. The canonical text on spinors and self-duality in Riemannian geometry is [AHS78]. Finally we should also say that our notation for spinors as applied to almost-Hermitian geometry comes from

[Sal91].

In the first two subsections we shall describe the spinor notation which we shall use throughout the rest of the thesis.

The final two subsections are on some aspects twistor theory relevant to almost-Kähler geometry and (anti)-self-dual almost-Kähler manifolds. The material in these sections is not used elsewhere in the thesis and our presentation is brief and discursive.

2.1.1 Representations of SO(4)

The representation theory of SU(2) is particularly simple. Let V denote the standard representation of SU(2) on $\mathbb{C}^2$. The SU(2) structure on $\mathbb{C}$ is given by a quaternionic structure $j : V \longrightarrow V$, i.e. an anti-holomorphic map with $j^2 = -1$ and a compatible Hermitian metric $g : V \otimes \overline{V} \longrightarrow \mathbb{C}$, together they define a complex symplectic form $\eta : V \otimes V \longrightarrow \mathbb{C}$. We see immediately from the existence of these invariant structures that $V \cong \overline{V} \cong V^*$. In fact the only irreducible representations of SU(2) are the symmetric powers $S^k V$. The only other ingredient we need is a way to decompose tensor products of these representations, this is given by the Clebsch–Gordon formula:

$$S^m V \otimes S^n V \cong \bigoplus_{k=0}^{\min\{m,n\}} S^{m+n-2k} V,$$

which arise by contracting repeatedly with the symplectic form. One gains immediately a good understanding of the representations of SO(4). If we write $\mathrm{Spin}(4) \cong \mathrm{SU}(2)^+ \times \mathrm{SU}(2)^-$, and write V^+ for the standard representation of $\mathrm{SU}(2)^+$ and V^- for the standard representation of $\mathrm{SU}(2)^-$ then the irreducible representations of Spin(4) are the representations $S^p V^+ \otimes S^q V^-$. Such a representation descends to a representation of SO(4) if and only if $p + q$ is even. Thus the irreducible representations of SO(4) can all be written $S^p V^+ \otimes S^q V^-$ (with $p + q$ even) and can be decomposed using the Clebsch–Gordon formula.

For example, $T^* M \cong [V^+ \otimes V^-]$. Another important example is $\bigwedge^\pm \cong$

$\mathfrak{su}(2)^{\pm} \cong S^2V^{\pm}$. Hence we have that

$$\begin{aligned}\mathcal{R} &\subseteq \bigwedge^2 \otimes \bigwedge^2 \\ &\cong (S^2V^+ \oplus S^2V^-) \otimes (S^2V^+ \oplus S^2V^-) \\ &\cong S^4V^+ \oplus S^2V^+ \oplus \mathbb{C} \oplus 2S^2V^+ \otimes S^2V^- \oplus S^4V^- \oplus S^2V^- \oplus \mathbb{C}\end{aligned}$$

Essentially just dimension counting shows that we must have $S^4V^+ \cong \mathcal{W}^+$, $S^4V^- \cong \mathcal{W}^-$ and $S^2V^+ \otimes S^2V^- \cong \mathcal{R}_0$. This in fact proves that the SO(4) decomposition of the curvature that we wrote down before was indeed a decomposition into irreducibles. To allow us to talk about vector bundles V^+ and V^- as well as just representations, we introduce:

Definition 2.1.1 *A* Spin(4) *structure on an oriented Riemannian* 4*-manifold, M, is a principal* Spin(4) *bundle P such that $P/\mathbb{Z}_2$ is the* SO(4) *bundle of oriented orthonormal frames.*

Associated to a Spin(4) structure on a 4-manifold, we have two 2-dimensional SU(2) bundles V^+, V^- together with an isomorphism $\phi : [V^+ \otimes V^-] \longrightarrow TM \otimes_{\mathbb{R}} \mathbb{C}$ which preserves the metrics. Conversely any two such bundles and isomorphism ϕ defines a Spin(4) structure. There are topological obstructions to the existence of Spin(4) structures, but they always exist locally. We shall mainly use them to prove local results and so we shall get in no trouble if we assume they exist.

We shall write $\eta^{\pm}$ for the symplectic form on $V^{\pm}$ and we shall write $\tilde{u}$ for the effect of the quaternionic structures $j^{\pm}$ of $V^{\pm}$ on $u \in V^{\pm}$.

2.1.2 Spinors and 4-dimensional almost Hermitian geometry

As well as elucidating the representation theory of SO(4), spinors can be used to understand U(2), and its relation to SO(4).

A choice of compatible almost-complex structure J on an oriented Riemannian 4-manifold with a Spin(4) structure corresponds to a choice of complex line $\langle u \rangle$ in V^+ by the condition that

$$\langle u \rangle \otimes V^- = \bigwedge^{1,0}.$$

Thus we can choose a section, u, of V^+ representing the choice of almost-complex structure. If we also insist that $\eta_+(u, \tilde{u}) = 1$, this u is determined at each point up to a factor $e^{i\theta}$. We shall refer to this choice as a choice of gauge.

Corresponding to the commutative diagram,

$$\begin{array}{ccc} \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) & \hookrightarrow & \mathrm{S}^1 \times \mathrm{SU}(2) \\ \downarrow & & \downarrow \\ \mathrm{SO}(4) & \hookrightarrow & \mathrm{U}(2) \end{array}$$

we see that the irreducible representations of U(2) are those of the form $\langle u^p \rangle \otimes S^q V^-$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and $p + q$ even (we are using product notation for symmetric products, and denoting $\tilde{u}^k$ by u^{-k}). Thus it is an easy matter to decompose representations by writing $S^k V^+$ as $\langle u^k \rangle \oplus \langle u^{k-2} \rangle \oplus \ldots \oplus \langle \tilde{u}^k \rangle$, and decomposing the $S^k V^-$ terms using the Clebsch–Gordon formula. For example $S^2 V^+ \cong \langle u^2 \rangle \oplus \mathbb{C} \oplus \langle \tilde{u}^2 \rangle \cong \bigwedge^{2,0} \oplus \mathbb{C} \oplus \bigwedge^{0,2}$.

We shall use such decompositions throughout the thesis without much comment.

As a simple example of the use of spinors in 4-dimensional almost Hermitian geometry, we review our material on the torsion in the language of spinors. Suppose that (M^4, g, J) is an almost-Hermitian manifold, and that $u \in V^+$ is some spinor representative of the almost-complex structure. The Levi–Civita connection ∇ induces a connection ∇ on $V^\pm$. So we can write:

$$\nabla u = \phi \otimes u + \psi \otimes \tilde{u}$$

for some ϕ, $\psi \in T^*M$. The condition that $\eta_+(u, \tilde{u}) = 1$ can easily be seen to imply that $\phi + \overline{\phi} = 0$. Note that under a change of gauge $u \longrightarrow e^{i\theta} u$, ϕ changes by the addition of $i\mathrm{d}\theta$. So we can, should we want to, always choose a gauge at any given point x such that $\phi_x = 0$.

On the other hand, although ψ is gauge dependent, one can readily see that it is related to the torsion tensor that we introduced earlier. $\overline{\nabla}$ lifts to give a connection on $V^\pm$ given by:

$$\overline{\nabla} u = \phi \otimes u$$

on V^+ and $\overline{\nabla} = \nabla$ on V^-. One calculates that:

$$\xi_X(u \otimes v) = \overline{\nabla}_X(u \otimes v) - \nabla_X(u \otimes v) = (-\psi, X)\tilde{u} \otimes v,$$

where $X \in TM$ and $v \in V^-$. Since ξ is real, we see that we must have:

$$\xi = 2\psi \otimes \tilde{u}^2 + 2\overline{\psi} \otimes u^2 \in [\![\bigwedge^{1,0}]\!] \otimes [\![\bigwedge^{2,0}]\!].$$

Hence if we write $\psi = \alpha \otimes u + \beta \otimes \tilde{u}$ with α, $\beta \in V^-$ then $\alpha \otimes u \otimes \tilde{u}^2 \in \langle \tilde{u} \rangle \otimes V^- \cong \bigwedge^{1,0}$ and $\beta \otimes \tilde{u} \otimes \tilde{u}^2 \in \langle \tilde{u}^3 \rangle \otimes V^-$ are gauge independent. So the two U(2) components of ξ are determined by α and β. In particular, M is Hermitian if and only if β vanishes and almost-Kähler if and only if α vanishes.

We shall often define gauge dependent tensors like α and β. If t is a gauge dependent tensor with the property that $t \otimes u^k$ is gauge independent then we shall say that t is gauge dependent of weight k (in this context, $u^{-1} := \tilde{u}$). Another way of saying this is that t corresponds to a well defined tensor in $\langle u^k \rangle \otimes S^n V^-$.

For example α has weight -1 and corresponds to a tensor in $\bigwedge^{0,1}$ and β has weight -3 and corresponds to a tensor in $\bigwedge^{0,1} \otimes \bigwedge^{0,2}$. For convenience we shall occasionally write $\hat{t}$ for the well defined tensor that t corresponds to.

On the other hand, ϕ is not a gauge dependent tensor of any weight (though it is of course gauge dependent). It is the connection form associated to our choice of gauge and as such it is only its curvature $\mathrm{d}\phi$ which has any invariant significance. We shall show in Section 3.3.1 that $\mathrm{d}\phi$ is related to the Ricci form of the manifold.

We shall make a slight alteration to our notation for the future. Since we are primarily interested in strictly almost-Kähler manifolds we shall usually have that $\alpha = 0$ and $\beta \neq 0$. In these circumstances we replace $\beta \in V^-$ by βv with $\beta \in \mathbb{R}^+$ and $v \in V^-$ of unit norm. So we have

$$\nabla u = \phi \otimes u + \beta v \otimes \tilde{u} \otimes \tilde{u}. \tag{2.1}$$

This change is, of course, purely cosmetic. However, it does emphasise that on a strictly almost-Kähler manifold we have, locally, an orthonormal basis for TM given by:

$$\{\Re(u \otimes v), \Re(iu \otimes v), \Re(u \otimes \tilde{v}), \Re(iu \otimes \tilde{v})\}$$

which is unique up to a choice of gauge. Thus our spinor notation combines the advantage of explicitness given by calculations in local coordinates with the advantage of making the representation theory of U(2) transparent. This is the reason for the power of the notation.

2.1.3 Twistor spaces

One of the most interesting phenomena related to (anti)-self-duality is the Twistor construction due to Penrose [Pen72] and interpreted by [AHS78].

If (M^4, g) is a Riemannian manifold then we shall denote the unit sphere bundle inside $\bigwedge^+$ by Z. Thus Z consists of all the almost-complex structures above each point, and an almost-complex structure compatible with the metric can be viewed simply as a section of Z. The Levi–Civita connection induces a horizontal distribution which we shall call $\mathcal{H}$. Thus we can decompose the tangent bundle of Z as $TZ \cong \mathcal{H} \oplus \mathcal{V} \cong TM \oplus TS^2$ where $\mathcal{V}$ is the tangent space of the fibre. Now $S^2 = \mathbb{C}P^1$ comes with a complex structure J' and a point $z \in Z$ above $x \in M$ can be viewed as a complex structure J_z on T_xM. Thus we can define two almost-complex structures on Z given by $J_1 = J_z \pm J'$ and $J_2 = J \mp J'$ (we shall choose the sign in a moment).

Any almost-complex structure, J, on M defines a section of Z. If the tangent plane to this section in Z is J_a invariant we shall say that J is J_a holomorphic. This gives the twistor interpretation of Hermitian and almost-Kähler structures on 4-manifolds:

Proposition 2.1.2 *[Sal84] One can fix the signs in the definition of J_1 and J_2 such that an almost-complex structure on M^4 is Hermitian iff it is J_1 invariant and almost-Kähler iff it is J_2 invariant.*

A natural question is whether or not the J_a's are ever integrable.

First suppose that J_2 can is integrable then, locally, there must be many J_2 holomorphic sections. Each of these sections would represent an almost-Kähler structure on M. However, as J_2 is integrable and the sections are J_2 holomorphic we would have that the sections would also represent integrable complex structures. Thus there would be many Kähler structures on M — far more than a two sphere's worth which is the most that is possible. A contradiction. So J_2 is never integrable.

Now suppose that J_1 is integrable. Again there must be many J_1 holomorphic sections each of which would represent a Hermitian structure. But recall that on a Hermitian manifold W^+_{00} vanishes. Hence we would have to have the W^+_{00} (defined with respect to *any* J_x) component of R vanishing.

This implies that $W^+ = 0$ in which case the manifold is called anti-self-dual (ASD). Thus J_1 is integrable only if the manifold is anti-self-dual.

In fact, the converse of the above result is true — it turns out that one can identify the Nijenhuis tensor of Z at a point J with W^+_{00}. For a proof see [AHS78]. To summarise:

Theorem 2.1.3 *[Pen76] (Z, J_1) is a complex manifold if and only if (M, g) is ASD.*

The components W^+ and W^- of the curvature tensor are conformally invariant as is the decomposition $\bigwedge^2 \cong \bigwedge^+ \oplus \bigwedge^-$. It is not hard to see that the construction of J_1 is also conformally invariant. We would like to be able to reverse this construction and obtain ASD conformal classes from complex manifolds. To do this one first needs to observe that we have a little more data: specifically the map $J \longrightarrow -J$ defines a free anti-holomorphic involution transforming each fibre to itself. One also should calculate that the normal bundle to the $S^2 \cong \mathbb{C}P^1$ fibres is isomorphic to $H \oplus H$ where H is the unique holomorphic line bundle over $\mathbb{C}P^1$ with $c_1(H) = 1$.

Theorem 2.1.4 *[Pen72] If Q is a complex 3-manifold which is fibred by $\mathbb{C}P^1$'s with normal bundle $H \oplus H$ and which possesses a free anti-holomorphic involution mapping each fibre to itself then Q is the twistor space of some manifold M with ASD conformal structure $[g]$.*

Thus anti-self-dual 4-manifolds can be given a completely holomorphic interpretation. Of course, one can give a similar interpretation to self-dual 4-manifolds just by reversing the orientation. This holomorphic interpretation of anti-self-duality is used in [Bes87] to construct hyperkähler metrics.

It looks, at first glance, as though we have no hope of giving a holomorphic interpretation of almost-Kähler structures — after all J_2 is never holomorphic. However, let us consider reverse oriented almost-Kähler structures on *ASD* four-manifolds.

Suppose (M^4, g) is an ASD manifold and that we have $\omega \in \bigwedge^-$ with $\mathrm{d}\omega = 0$. Since the ASD condition is conformally invariant, we can, at points where $\omega \neq 0$, rescale the metric to ensure that $\|\omega\| = 1$ and hence ensure that ω describes a reverse-oriented almost-Kähler structure. So if we have any ASD manifold and any non-vanishing closed form $\omega \in \bigwedge^-$ then we effectively have a reverse-oriented almost-Kähler, ASD manifold.

Now since $\mathrm{d}\omega = 0$, we can, at least locally, write $\omega = \mathrm{d}\eta$ for some η in $\bigwedge^1$. We may consider η as the connection form of some connection on the trivial bundle over M. Correspondingly η is well defined up to gauge transformations. Since $\mathrm{d}\eta = \omega \in \bigwedge^-$ we see that the connection is itself anti-self-dual.

We make this rather circuitous reinterpretation of a closed $\bigwedge^-$ form because we wish to show how our situation fits in with Ward's correspondence between anti-self-dual connections on ASD manifolds and holomorphic bundles over the twistor space [AHS78]. In the present context this gives:

Lemma 2.1.5 *If $(M^4, [g])$ is a manifold with an anti-self-dual conformal class, then reverse oriented almost-Kähler structures on M compatible with $[g]$ are in one to one correspondence with closed, non-vanishing, $\bigwedge^{1,1}$ forms, η on Z which vanish on the $\mathbb{C}P^1$ fibres and which satisfy $\tau^*\eta = -\eta$.*

The correspondence is given simply by taking η to be the pull-back $\pi^*\omega$ of ω.

Of course, we can reverse orientations to obtain a holomorphic interpretation of almost-Kähler, self-dual manifolds.

As well has having canonical almost-complex structures, the twistor space of a Riemannian manifold has canonical metrics — one takes the original metric on $\mathcal{H} \cong TM$ and takes a multiple λ of the round metric on $\mathcal{V} \cong S^2$. Some of the metrics obtained in this way are interesting. For example:

Theorem 2.1.6 *If M is an ASD, Einstein manifold then*

- *if the scalar curvature is positive, then for suitable λ the metric on Z is Kähler Einstein.*
- *if the scalar curvature is negative, then for suitable λ the metric on Z is almost-Kähler with J-invariant Ricci tensor (that is $Ric(X,Y) = Ric(JX,JY)$ for all X, $Y \in TM$).*

In fact the only compact possibilities for M with positive scalar curvature are $\mathbb{C}P^2$, and S^4 with their standard metrics. The above result is a first step in the original proof of this fact given in [Hit81].

The only known compact possibilities for M with negative scalar curvature are quotients of hyperbolic space and complex hyperbolic space $\mathbb{C}H^2$. The second part of the above theorem was found in [DM90] and provides a counterexample to the conjecture made in [BI86] that a compact almost-Kähler manifold with J-invariant Ricci tensor is necessarily Kähler.

2.1.4 Self-dual and anti-self-dual almost-Kähler manifolds

Although the main object of study in this thesis is Einstein almost-Kähler manifolds, it seems appropriate now to include a brief discussion of (anti)-self-dual almost-Kähler manifolds.

To place our discussion in context, we notice that a Kähler four-manifold is anti-self-dual if and only if it is scalar flat. These scalar-flat Kähler manifolds have been considered extensively and are rather well understood — see [LeB91a], [LS93], [KP95] amongst others. Self-dual Kähler 4-manifolds are also well understood: compact, self-dual Kähler 4-manifolds are either conformally flat, a quotient of complex hyperbolic space $\mathbb{C}H^2$ with its standard metric or $\mathbb{C}P^2$ with the Fubini–Study metric.

As we observed in the previous section, the fact that W^+ and $\bigwedge^+$ are conformally invariant means that we can interpret ASD almost-Kähler manifolds simply as 4-manifolds M with an ASD conformal structure $[g]$ and a nowhere vanishing closed form ω in $\bigwedge^+$. Since $\omega \in \bigwedge^+$ the condition that it is closed is equivalent to the condition that it is harmonic. This yields a simple construction of compact almost-Kähler ASD manifolds:

Take a scalar flat Kähler manifold and deform its metric slightly within the moduli space of ASD metrics. A small deformation of the metric results in a small deformation of $\mathcal{H}^+$, the space of harmonic forms in $\bigwedge^+$ (use the ellipticity of the Laplacian to prove this). Thus the form $\omega \in \bigwedge^+$ can be deformed to a new form ω' which will be non-vanishing if our deformation of the metric is small enough and hence define a new almost-Kähler structure.

In other words, if $\mathcal{M}_{ASD}$ is the moduli space of ASD structures on our manifold M and if $[g]$ is a regular point of $\mathcal{M}_{ASD}$ and admits an almost-Kähler structure then so does any metric in $\mathcal{M}_{ASD}$ suitably near M.

Similar remarks apply to the self-dual case.

As an alternative to asking whether or not an ASD metric admits a compatible almost-Kähler structure, one could ask when a given symplectic manifold (M^4, ω) admits a compatible ASD metric. Thus we are interested in the moduli space $\mathcal{M}^{\omega}_{ASD}$ of ASD metrics compatible with ω. Our discussion mimics that in [KK92], where the moduli space of ASD metrics is considered. Thus we shall be very brief.

We consider two metrics to be the same if they are mapped to each other by a symplectomorphism. Recall that the identity component of the group of symplectomorphisms is given by the Hamiltonian vector fields — which are determined by taking d of any function and then using the isomorphism between T^*M and TM defined by ω. Also the tangent space at g to the space of metrics compatible with a given ω is given by $[\![S^{2,0}]\!] \cong [\![S^2(\bigwedge^{1,0})]\!] \cong [\![\langle u^2 \rangle \otimes S^2 V^-]\!]$. Thus associated to our moduli problem we have the deformation complex:

$$\Gamma(\mathbb{R}) \longrightarrow \Gamma([\![S^{2,0}]\!]) \longrightarrow \Gamma(\mathcal{W}^+).$$

If one counts dimensions, one sees immediately that this complex stands every chance of being elliptic. It is not at all hard to check that it is. The Atiyah–Singer index theorem as stated in ([AS68] Proposition 2.17) allows us to compute the indices of these complexes as a calculation using Chern characters. One finds that the virtual dimension of the moduli space of ASD metrics compatible with ω is $14\tau + 8\chi$.

One could consider the parallel problem for self-dual structures. One now considers the deformation complex:

$$\Gamma(\mathbb{R}) \longrightarrow \Gamma([\![S^{2,0}]\!]) \longrightarrow \Gamma(\mathcal{W}^-).$$

Once again this is elliptic. In summary:

Lemma 2.1.7 *The virtual dimension of the moduli space of ASD metrics compatible with ω is $14\tau + 8\chi$.*

The virtual dimension of the moduli space of SD metrics compatible with ω is $-15\tau + 8\chi$.

By way of comparison the virtual dimension of the moduli space of ASD metrics (not necessarily compatible with ω) is $\frac{1}{2}(15\chi + 29\tau)$. Of course, this virtual dimension will only be equal to the dimension of the moduli space when the gauge group (of symplectomorphisms) acts freely and when the complex is unobstructed — i.e. H^2 of the complex is zero.

In conclusion we see that the problem of finding (A)SD almost-Kähler manifolds is determined by an elliptic complex. As such one might hope that the problem is amenable to methods from functional analysis. Since it is easy to find (A)SD, strictly almost-Kähler structures on manifolds which admit (A)SD Kähler structures, we pose:

Question 2.1.8 *Do there exist compact* 4*-manifolds which admit an almost-Kähler (A)SD structure but do not admit any Kähler (A)SD structure?*

The author does not know the answer to this question, but conjectures that, at least in the ASD case that the answer is yes. To justify this we outline a program one could follow in an attempt to construct examples.

The first observation is that the ASD condition would appear to behave well under blowing up (i.e. connect summing with $\overline{\mathbb{C}P^2}$, reverse oriented projective space which is itself ASD). For example Floer was able to show using analysis in [Flo91] that $\overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}\#\ldots\#\overline{\mathbb{C}P^2}$ always admits ASD metrics. LeBrun went on in [LeB91b] to construct such metrics explicitly. Generalizing Floer's results, Donaldson and Friedman were able to prove, in [DF89], using twistor methods and complex deformation theory, that any ASD manifold with H^2 vanishing can be blown up to yield new ASD manifolds. Taubes managed to go yet further, and proved (using functional analysis) that:

Theorem 2.1.9 *[Tau92] If one blows up any compact four manifold sufficiently often, the manifold one obtains will admit an ASD metric.*

Secondly, the symplectic condition behaves well under blow-ups — see for example [MS95] for the symplectic blow-up construction.

Thirdly, the local model one uses both for the symplectic blow-up construction and the gluing procedure in the analytical proofs of the existence of ASD structures is the Burns metric (see for example [LeB91b]) with its standard Kähler form.

Thus one strongly suspects that one should be able to prove:

Conjecture 2.1.10 *If (M^4, g, ω) is compact, ASD, almost-Kähler and if we have $H^2 = 0$ then $M \# \overline{\mathbb{C}P^2}$ admits an almost-Kähler structure.*

This does not immediately help with answering our question, however, since the only examples of compact ASD, almost-Kähler manifolds we could apply it to all admit scalar flat Kähler structures. One has:

Theorem 2.1.11 *[LS93] [KP95] If (M^4, g, ω) is compact, scalar flat Kähler and $H^2 = 0$ then $M \# \overline{\mathbb{C}P^2}$ admits a scalar flat Kähler structure.*

However, in the case when $H^2 \neq 0$ Donaldson and Friedman were able to show how one can get around the obstruction by blowing-up sufficiently often. This is not possible in the scalar flat Kähler case. For example the $K3$ surface is scalar flat Kähler but none of its blow-ups admit scalar flat Kähler metrics (see [LeB91a]). Since we have additional flexibility about how we attach our $\overline{\mathbb{C}P^2}$ in the almost-Kähler case compared to the scalar flat Kähler case, one may hope that one could construct almost-Kähler, ASD metrics on blow ups of the $K3$ surface.

The author has not been able to get very far with the analysis required to prove any of these results — the papers [Tau92] and [Flo91] are not particularly closely adapted to our problems. On the other hand one cannot, unfortunately, attempt a twistor proof since one does not have a twistor interpretation of ASD almost-Kähler structures. Whilst we do have a twistor interpretation of SD Kähler structures, our formulae for the dimension of the moduli space suggests that SD structures will behave badly under blowing up, whereas as is shown in [MS95], symplectic structures behave very badly on connect summing with $\mathbb{C}P^2$.

2.2 Integral formulae

2.2.1 The Hitchin–Thorpe inequality

One important feature of four dimensions that is particularly relevant to this thesis is that no topological obstructions are known to the existence of Einstein metrics on a given manifold in dimensions higher than four. The two dimensional case is well understood — but of no real interest to us since any oriented Riemannian 2-manifold is Kähler. The three dimensional case is extremely interesting and increasingly well understood due to Thurston's deep results. However, since we are primarily interested in almost-Hermitian geometry, we shall not consider odd dimensional manifolds.

The known topological obstructions to the existence of Einstein metrics on four manifolds are all variants on the Hitchin–Thorpe inequality which we shall now describe. We recall that the curvature of our 4-manifold, M, is given by:

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12}\mathbf{1} & R_0 \\ \hline R_0 & W^- + \frac{s}{12}\mathbf{1} \end{array} \right).$$

Now by the Chern–Weil theorem, the first Pontrjagin number p_1 of our manifold is given by integrating an appropriate quadratic function of the curvature over the manifold. Recall that Schur's lemma can be rephrased as: if V, W are irreducible $G \subseteq \mathrm{SO}(n)$ modules and if $\tau : V \otimes W \longrightarrow \mathbb{R}$ is an equivariant, bilinear map then either $V \cong W$, in which case τ is a multiple of the metric on V, or $V \ncong W$, in which case τ is zero.

We conclude that:

$$p_1 = \int_M (*\|W^+\|^2 + *\|W^-\|^2 + *s^2 + *\|R_0\|^2)$$

where the $*$'s are some constants which can easily be identified. By the Hirzebruch signature theorem $\tau = \frac{1}{3}p_1$ where τ is the signature of the manifold. Thus we have an integral formula for τ. Similarly, the Gauss Bonnet theorem tells us that the Euler characteristic χ is given by an integral of the same form. The end result is:

$$\chi = \frac{1}{8\pi^2} \int_M (\|W^+\|^2 + \|W^-\|^2 + \frac{1}{24}s^2 - \frac{1}{2}\|R_0\|^2)$$

and

$$\tau = \frac{1}{12\pi^2} \int_M (\|W^+\|^2 - \|W^-\|^2).$$

Now, if M is Einstein we have that $R_0 \equiv 0$. Hence one can calculate that:

$$2\chi + 3\tau = \frac{1}{4\pi^2} \int_M (2\|W^+\|^2 + \frac{1}{24}s^2) \geq 0.$$

We have immediately:

Theorem 2.2.1 *[Hit74] [Tho69] If (M^4, g) is a compact, oriented, Einstein 4-manifold then $2\chi + 3\tau \geq 0$ with equality if and only if $W^+ \equiv 0$ and $s \equiv 0$ — i.e. if and only if the manifold is hyperkähler. Similarly $2\chi - 3\tau \geq 0$.*

This result is the basic non-existence result for Einstein metrics on 4-dimensional manifolds. As we shall see it has a number of interesting refinements.

The condition that M is four-dimensional is clearly crucial to the proof above result. Firstly self-duality is obviously important and this phenomenon is special to four dimensions. Secondly the fact that we integrate a quadratic is essential to the use of Schur's Lemma and thence to the simple form of the integrands. Of course, it is conceivable that one could prove similar inequalities in the characteristic numbers of higher dimensional manifolds by the same type of algebraic argument. However, it is shown in [BP81] that there exist algebraic counter examples to any such generalisation. Although changing the dimension does not allow one to generalise this result, one can do so by weakening the Einstein condition. For example, the case of four-dimensional Einstein Weyl manifolds has been examined in [SP94].

There are a number of interesting refinements to the Hitchin–Thorpe inequality.

Firstly, in the case when $2\chi \pm 3\tau = 0$, Hitchin showed in [Hit81] that any Einstein manifold must be a quotient of T^4 with the flat metric or a quotient of the $K3$ surface equipped with a Calabi–Yau metric.

A second way of refining the Hitchin–Thorpe inequality is by considering the *Gromov simplicial volume*. This is defined as follows:

The fundamental class, $[M]$, of a compact, oriented n dimensional manifold is a singular homology class in $H_n(M, \mathbb{R})$. That is to say it is an equivalence class of linear combinations of simplices $c = \sum \lambda_i \sigma_i$ with $\lambda_i \in \mathbb{R}$ and $\partial c = 0$. The simplicial volume $\|M\|$ is defined to be the infimum of all sums $\sum |\lambda_i|$ as one runs through all possible representatives of $[M]$.

The simplicial volume is clearly a homotopy invariant of the manifold. In fact Gromov shows in [Gro82] that it only depends on a certain representation of the fundamental group of the manifold. In particular it vanishes on simply connected manifolds. The relevance of the simplicial volume to the study of Einstein metrics comes from the following theorem.

Theorem 2.2.2 (Gromov's main inequality) *[Gro82] If (M, g) is compact with Ric $\geq -(n-1)g$, then the volume V of M satisfies $V \geq C\|M\|$ where C is a constant depending only on the dimension. In dimension 4 one may take $C = \frac{1}{1944}$.*

Corollary 2.2.3 *[Gro82] Let M be a 4-dimensional compact manifold then M admits an Einstein metric only if $\|M\| \leq 2592\pi^2\chi$*

Proof: Suppose there is an Einstein metric with negative scalar curvature. We rescale the metric so that $r = -(n-1)g$. We can now use the integral formula for χ to see that $\chi \geq \frac{3}{4\pi^2}V$. □

Of course, one can compare Gromov's estimate for the volume with different linear combinations of χ and τ to obtain similar results — and indeed many of the integral formulae on Einstein manifolds can be sharpened by using Gromov's estimate. Some of the integral formulae that we shall obtain for almost-Kähler manifolds can be sharpened in this way. We do not do so explicitly in this thesis because the author does not know of any interesting applications.

Gromov's ideas have been followed up by a number of authors to prove the non-existence of Einstein metrics on certain manifolds with "large" fundamental group. For example it is proved in [BCG94] that every Einstein metric on a compact quotient of hyperbolic space is isometric to its standard constant curvature metric.

One further way in which the Hitchin–Thorpe inequality can be refined is by consideration of the Seiberg–Witten invariants. We shall postpone discussion of this until section 2.3.

2.2.2 An application to almost-Kähler, Einstein manifolds

A simple observation allows us to connect the Hitchin–Thorpe inequality with almost-Kähler geometry almost immediately. Suppose that (M^4, g, J) is almost-Kähler and that the torsion ξ is nowhere vanishing then since $\xi \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$, we must have that the Euler class $e([\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!])$ of this bundle must be zero. Using Chern characters, one can easily compute the Chern classes of a tensor product of bundles — [Ati67]. One easily finds that $e([\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]) = 5\chi + 6\tau$.

Theorem 2.2.4 *If (M^4, g, J) is a compact almost-Kähler, Einstein 4-manifold then ξ must vanish at at least one point of M.*

Proof: *Suppose that ξ does not vanish anywhere; then $5\chi + 6\tau = 0$. On the other hand $2\chi + 3\tau \geq 0$. Combining these facts one must have that $\chi \leq 0$. On the other hand it follows straight away from the integral formula for χ that $\chi \geq 0$ with equality if and only if the manifold is flat. So $\chi = 0$ and the manifold is flat. But we proved in the previous chapter that an almost Kähler structure compatible with the flat metric is necessarily Kähler, in which case ξ vanishes identically. A contradiction.* $\square$

Given the fact that $\|\xi\|^2 = \frac{s^* - s}{16}$ on an almost Kähler 4-manifold and the fact that the scalar curvature is constant on an Einstein manifold, we can state:

Corollary 2.2.5 *If (M^4, g, J) is a compact almost-Kähler, Einstein 4-manifold with constant $*$-scalar curvature then it is Kähler.*

This, of course, covers the case of strongly $*$-Einstein, Einstein 4-manifolds which have been the object of a certain amount of study, in [SV90] for example. Since the $*$-scalar curvature is calculated from a component W^+ and the scalar curvature, we immediately have:

Corollary 2.2.6 *If (M^4, g, J) is compact, anti-self-dual (i.e. $W^+ \equiv 0$), Einstein and almost-Kähler then it is necessarily Kähler.*

Of course, the condition that the manifold is anti-self-dual and Einstein is an algebraically strong condition. It is perhaps not surprising then that one can still prove the result if one drops the compactness condition. We shall prove this in Chapter 3. However, the proof is much more intricate than the one we have just given for the compact case. It is also interesting to note that Tod's examples show that strictly almost-Kähler, Einstein, self-dual 4-manifolds do exist locally.

Notice that since the Einstein metric on quotients of hyperbolic space are unique [BCG94], we have the corollary that quotients of hyperbolic space do not admit almost-Kähler Einstein metrics.

2.2.3 The curvature of $\overline{\nabla}$

Suppose now that (M^4, g, J) is a compact, almost-Kähler, Einstein 4-manifold. As well as the Levi–Civita connection, ∇, we have the unitary connection $\overline{\nabla}$. By the Chern-Weil theorem, one could equally well evaluate p_1 or χ using the curvature $\overline{R}$ of $\overline{\nabla}$. Schur's lemma will once again ensure that the formulae take a simple form. To do this we shall first need to evaluate $\overline{R}$.

Lemma 2.2.7 *If $\overline{R}$ denotes the curvature associated to $\overline{\nabla}$ then*

$$\overline{R}(X,Y)Z = R(X,Y)Z - [\xi_X, \xi_Y]Z + (\overline{\nabla}_{[X}\xi)_{Y]}Z + \xi_{\xi_X Y - \xi_Y X}Z. \qquad (2.2)$$

Proof:

$$\begin{aligned}
\overline{R}(X,Y)Z &= (\overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X)Z - \overline{\nabla}_{[X,Y]}Z \\
&= \overline{\nabla}_X\nabla_Y + (\overline{\nabla}_X\xi)_Y Z + \xi_{\overline{\nabla}_X Y}Z + \xi_Y\overline{\nabla}_X Z \\
&\quad + \textit{the same with } X \textit{ and } Y \textit{ reversed} \\
&\quad -\nabla_{[X,Y]}Z - \xi_{[X,Y]}Z \\
&= \nabla_X\nabla_Y Z + \xi_X\nabla_Y Z + (\overline{\nabla}_X\xi)_Y Z + \xi_{\overline{\nabla}_X Y}Z + \xi_Y\overline{\nabla}_X Z \\
&\quad + \textit{the same with } X \textit{ and } Y \textit{ reversed} \\
&\quad -\nabla_{[X,Y]}Z - \xi_{[X,Y]}Z \\
&= \nabla_X\nabla_Y Z + \xi_X\overline{\nabla}_Y Z - \xi_X\xi_Y Z + (\overline{\nabla}_X\xi)_Y Z \\
&\quad + \xi_{\overline{\nabla}_X Y}Z + \xi_Y\overline{\nabla}_X Z \\
&\quad + \textit{the same with } X \textit{ and } Y \textit{ reversed} \\
&\quad -\nabla_{[X,Y]}Z - \xi_{[X,Y]}Z \\
&= R(X,Y)Z - [\xi_X, \xi_Y]Z + (\overline{\nabla}_{[X}\xi)_{Y]}Z + \xi_{\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]}Z.
\end{aligned}$$

Since ∇ is torsion free one shows that the torsion of $\overline{\nabla}$ is given by

$$\begin{aligned}
\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] &= \nabla_X Y + \xi_X Y - \nabla_Y X - \xi_Y X - [X,Y] \\
&= \xi_X Y - \xi_Y X.
\end{aligned}$$

This completes the proof. □

To make this formula slightly more digestible we shall make some new definitions. Let V be any representation of $\mathrm{U}(n)$. Define

$$\alpha : \bigwedge^1 \otimes \bigwedge^1 \otimes V \longrightarrow \bigwedge^2 \otimes V$$

by antisymmetrisation on the first two factors. Define

$$\beta : (\bigwedge^1 \otimes \mathfrak{u}(n)^\perp) \odot (\bigwedge^1 \otimes \mathfrak{u}(n)^\perp) \longrightarrow \bigwedge^1 \otimes \bigwedge^1 \otimes \mathfrak{u}(n)^\perp$$

by contracting the first $\mathfrak{u}(n)^\perp$ with the second $\bigwedge^1$. Finally we define

$$\gamma : (\bigwedge^1 \otimes \mathfrak{u}(n)^\perp) \odot (\bigwedge^1 \otimes \mathfrak{u}(n)^\perp) \longrightarrow \bigwedge^1 \otimes \bigwedge^1 \otimes \mathfrak{gl}_n\mathbb{R}$$

by multiplication of the endomorphisms.

So we have that

$$\overline{R}(X,Y) = R(X,Y) - 2\alpha \circ \gamma(\xi \odot \xi) - 2\alpha \circ \beta(\xi \odot \xi) + 2\alpha(\overline{\nabla}\xi). \qquad (2.3)$$

Lemma 2.2.8 $\alpha \circ \gamma : \bigodot^2 \left(\bigwedge^1 \otimes \mathfrak{u}(n)^\perp\right) \longrightarrow \bigwedge^2 \otimes \mathfrak{u}(n)$.

Proof: *First note that*

$$\alpha \circ \gamma : \bigodot^2 \left(\bigwedge^1 \otimes \mathfrak{u}(n)^\perp\right) \longrightarrow \bigwedge^2 \mathfrak{so}(n)$$

since $(\alpha \circ \gamma(\xi \odot \xi))(X,Y) = \xi_X \xi_Y - \xi_Y \xi_X$ *and both* ξ_X *and* ξ_Y *are in* $\mathfrak{so}(n)$.

The result now follows by Schur's lemma since $\mathfrak{u}(n)^\perp \otimes \mathfrak{u}(n)^\perp$ *and* $\mathfrak{u}(n)^\perp$ *have no irreducible components in common [FFS94].* □

The spaces where the other terms of equation(2.3) reside are obvious — if one remembers that $\overline{\nabla}$ is unitary. The information is summed up below:

$$\underbrace{\overline{R}(X,Y)}_{\bigwedge^2 \otimes \mathfrak{u}(n)} = \underbrace{R(X,Y)}_{\mathcal{R}} - \underbrace{2\alpha \circ \gamma(\xi \odot \xi)}_{\bigwedge^2 \otimes \mathfrak{u}(n)} - \underbrace{2\alpha \circ \beta(\xi \odot \xi)}_{\bigwedge^2 \otimes \mathfrak{u}(n)^\perp} + \underbrace{2\alpha(\overline{\nabla}\xi)}_{\bigwedge^2 \otimes \mathfrak{u}(n)^\perp}. \qquad (2.4)$$

In pictorial form we have that:

$$\overline{R} = \left(\begin{array}{c|c||c} a & W_F^+ & 0 \\ \hline 0 & 0 & 0 \\ \hline\hline 0 & 0 & W^- + \frac{a+b}{3}\mathbf{1} \end{array}\right) + \rho(\xi \odot \xi)$$

where ρ is some equivariant map.

To complete our analysis of our equation for R it is convenient to choose a basis as in section 1.2.3. If one writes $\overline{R}$ in block form with respect to this basis as in Lemma 1.2.5 and recalls that $b = \|\xi\|^2$ one finds that

$$\overline{R} = \left(\begin{array}{c|c||ccc} a + \frac{1}{2}b & W^+F & \frac{1}{2}b & 0 & 0 \\ \hline 0 & 0 & \multicolumn{3}{c}{0} \\ \hline\hline 0 & 0 & \multicolumn{3}{c}{W^- + \frac{s}{12}} \end{array} \right).$$

2.2.4 Sekigawa's integral formula

It is now clear that the formula one obtains by applying the Chern-Weil theorem to $\overline{R}$ takes the form:

$$\tau = \int_M (*a^2 + *ab + *b^2 + *\|W^+F\|^2 + *\|W^-\|^2).$$

This is for the most part a consequence of Schur's lemma. The $\bigwedge^-$ component of $\overline{R}$ is determined by $\xi \odot \xi$ and one expects a term proportional to the norm of this in our integral formula. In our case this is equal to b^2 which is proportional to $\|\xi\|^4$. This is clear from our explicit formula for $\overline{R}$ with respect to an appropriate basis. A more invariant proof is the observation that U(2) acts transitively on the unit sphere in $[\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$. Thus the only possible (non-linear) equivariant maps from $[\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ to $\mathbb{R}$ are functions of $\|\xi\|$. Thus the contribution from the $\bigwedge^-$ term of $\overline{R}$ to the integral formula must be proportional to $\|\xi\|^4$.

It is a simple matter to identify the constants. The net result is:

$$\tau = \frac{1}{12\pi^2} \int_M \left(\frac{2}{3}a^2 + \frac{1}{3}ab - \frac{1}{3}b^2 + \|W^+_F\|^2 - \|W^-\|^2 \right).$$

On the other hand our original formula for τ can be written as:

$$\tau = \frac{1}{12\pi^2} \int_M \left(2\|W^+_{00}\|^2 + \frac{1}{6}(2a-b)^2 + 2\|W^+_F\|^2 - \|W^-\|^2 \right)$$

when one decomposes $\|W^+\|^2$. Taking the difference of these two formulae one sees that on a compact, almost-Kähler, Einstein 4-manifold:

$$\int_M (4\|W^+_{00}\|^2 + 2\|W^+_F\|^2 + b^2 - 2ab) = 0,$$

an integral formula first due to Sekigawa ([Sek85]).

One is naturally tempted to try the same trick with χ, but this in fact gives exactly the same formula.

As a consequence of this formula we obtain:

Theorem 2.2.9 *If (M^4, g, J) is a compact, strictly almost-Kähler, Einstein manifold then the scalar curvature is negative.*

Proof: *One sees immediately that*

$$\int_M ((a+b)b) \geq 0$$

with equality if and only if the manifold is Kähler. Since b is everywhere non-negative we must have that $a+b$ is positive somewhere. But $s = -\frac{a+b}{4}$. Hence the scalar curvature must be negative somewhere. But the scalar curvature is constant. $\square$

One immediate corollary of this result is that a strictly almost-Kähler Einstein manifold must satisfy the strict inequality $2\chi \pm 3\tau > 0$.

Remark: *In actual fact, Sekigawa generalises this result to all dimensions in [Sek87]. The way in which he does this is to consider the formulae for $p_1 \wedge \omega^{n-2}$ obtained from the Chern-Weil theorem using R and then $\overline{R}$. Once again Schur's lemma guarantees that the formulae will consist mostly of sums of squares. However, $\mathrm{U}(n)$ does not act transitively on $\mathcal{W}_2$ when $n \geq 0$ and thus one obtains a quartic term in ξ which is somewhat harder to deal with. Thus the proof is substantially more complex.*

Similarly, one is tempted to try to find inequalities between $c_1^2 \wedge \omega^{n-2}$ and $c_2 \wedge \omega^{n-2}$ analogous to the Hitchin–Thorpe inequality on higher dimensional almost-Kähler manifolds. To the author's knowledge, no-one has succeeded in doing this although the author knows of a number of people who have tried. Again it is the quartic terms that make the algebra difficult.

Remark: *Another route to proving Sekigawa's formulae is to derive Weitzenböck formulae for the action of ∇ on ξ. We shall see these Weitzenböck formulae in Chapter 4. Yet another method is to perform a very direct analysis of the differential Bianchi identity. These methods have the advantage* *of proving the relevant local formulae but are rather more tedious to carry out.*

Two further applications of our integral formulae should be mentioned. Firstly one can prove:

Theorem 2.2.10 *[LeB95a] On a compact almost-Kähler, Einstein 4-manifold we have that $\chi \geq 3\tau$.*

This result was first proved by combining deep results of Taubes and LeBrun on the Seiberg–Witten invariants. However, as was observed by [Dra97], one can give an elementary proof by simply combining our integral formulae. As we shall see when we discuss the Seiberg–Witten invariants later, the proof using Seiberg–Witten theory shows that this result does not depend on the *simultaneous* existence of a symplectic structure and Einstein metric and thus it will turn out that the almost-Kähler geometry is not important.

Another result which one can prove using the integral formulae is:

Proposition 2.2.11 *[Dra97] If (M^4, g, J) is a compact, strictly, almost-Kähler, Einstein 4-manifold with first Chern class $c_1(M)$ then $c_1 \wedge [\omega] > 0$, where $[\omega]$ is the de Rham cohomology class represented by ω.*

Although this result at present only gives a "symplectic topological" obstruction to the existence of almost-Kähler, Einstein metrics, we shall be able to combine it with results on the Seiberg–Witten invariants to give topological obstructions to the existence of almost-Kähler, Einstein metrics.

2.3 The Seiberg–Witten equations

The Seiberg–Witten invariants are diffeomorphism invariants of 4-dimensional manifolds. A major reason for their interest is that they are *not* homeomorphism invariants. They provide a powerful tool for showing that the classification of smooth, simply connected 4-manifolds up to diffeomorphism is wildly different from the corresponding classification of simply connected 4-manifolds up to homeomorphism obtained by Freedman. Results showing this disparity, which is unique to dimension 4, were first obtained by Donaldson in [Don83] using the self-dual Yang Mills equations and there is a large literature on the subject. The interested reader would do well to consult [DK90]. The Seiberg–Witten equations were introduced by Witten in [Wit94] where it was shown how these invariants can give simpler proofs of results previously obtained by using Donaldson theory. A physical argument for the essential equivalence of the theories is also given. Subsequently many authors have gone on to prove results on differential topology using the Seiberg–Witten equations that seemed too difficult to obtain using Donaldson theory. A survey is given in [Don96].

More to the point for this thesis, as well as revolutionising the study of differential topology in 4-dimensions, the Seiberg–Witten equations have revolutionised the study of symplectic topology in that dimension. Furthermore new topological obstructions to the existence of Einstein metrics in 4-dimensions have been found by studying the Seiberg–Witten invariants. Our aim in this section is to discuss these developments and examine what light they shed on almost-Kähler geometry.

2.3.1 Algebraic preliminaries

The group $\mathrm{Spin}^c(4)$ is defined by

$$\mathrm{Spin}^c(4) \cong \mathrm{Spin}^c(4) \times_{\mathbb{Z}^2} \mathrm{S}^1.$$

One has immediately two useful exact sequences

$$0 \longrightarrow \mathrm{S}^1 \longrightarrow \mathrm{Spin}^c(4) \longrightarrow \mathrm{SO}(4) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{SO}(4) \longrightarrow \mathrm{Spin}^c(4) \longrightarrow \mathrm{S}^1 \longrightarrow 0.$$

A $\mathrm{Spin}^{\mathrm{c}}$ structure on an oriented Riemannian 4-manifold (M, g) is a principal $\mathrm{Spin}^{\mathrm{c}}$ bundle over M with the property that the projection to $\mathrm{SO}(4)$ induces the bundle of oriented orthonormal frames.

The projection to S^1 in the second sequence defines a principal S^1 bundle over the manifold. Using the standard action of $\mathrm{S}^1 \cong \mathrm{U}(1)$ on $\mathbb{C}$, we obtain a Hermitian line bundle L over our manifold.

We shall be interested in the representation theory of Spin^c. To understand this we take a local lifting of Spin^c to $\mathrm{Spin}(4) \times \mathrm{S}^1$. This allows us to define bundles V^+, V^- and $L^{\frac{1}{2}}$ associated to the fundamental representations of

Spin(4) and S^1. The irreducibles of Spin(4) $\times$ S^1 are the spaces $S^m V^+ \otimes S^n V^- \otimes L^{\frac{p}{2}}$. Such a space represents a bona-fide Spin^c bundle if and only if $m+n+p$ is even. Once again the Clebsch–Gordon formula allows us to decompose tensor products of these irreducible representations.

It is standard to write $W^\pm$ for the basic Spin^c representations defined by $V^\pm \otimes L^{\frac{1}{2}}$.

A Spin^c connection is clearly determined uniquely by a connection on the SO(4) bundle of oriented orthonormal frames and an S^1 connection on the line bundle L. One has a natural connection, the Levi–Civita connection, on the bundle of oriented orthonormal frames. So to each unitary connection A on L we can associate a Spin^c connection. We shall write ∇_A for the corresponding covariant derivatives. We shall write $F_A \in \bigwedge^2 \otimes \mathfrak{s}^1 \cong \bigwedge^2 \otimes i\mathbb{R}$ for the curvature of A.

If $\Phi \in W^\pm \cong V^\pm \otimes L^{\frac{1}{2}}$ then

$$\nabla_A \Phi \in T^* M \otimes W^\pm \cong S^2 V^\pm \otimes V^\mp \otimes L^{\frac{1}{2}} \oplus V^\mp \otimes L^{\frac{1}{2}}.$$

We can then define the (twisted) Dirac operator $D_A : \Gamma(W^\pm) \longrightarrow \Gamma(W^\mp)$ by sending Φ to the component of $\nabla_A \Phi$ in $V^\mp \otimes L^{\frac{1}{2}} \cong W^\mp$. Since D_A is (up to scale) the only first order operator from $W^\pm$ to $W^\mp$ given by composing ∇_A with an equivariant map, we see that D_A must be essentially self-adjoint.

Of fundamental importance in the study of the Dirac operator is the following Weitzenböck formula.

Theorem 2.3.1 *If $\Phi \in W^+$ then*

$$D_A^2 \Phi = \nabla_A^* \nabla_A \Phi + \frac{s}{4}\Phi + \frac{F_A^+}{2}\Phi$$

where we view $F_A^+ \in S^2 V^+$ as an endomorphism of W^+ via the identification $\mathrm{End}(W^+) \cong (V^+ \otimes L^{\frac{1}{2}}) \otimes (V^+ \otimes L^{-\frac{1}{2}}) \cong S^2 V^+ \oplus \mathbb{C}$.

Proof: *If one decomposes $T^* \otimes T^* \otimes V^+ \otimes L^{\frac{1}{2}}$ into irreducibles using the Clebsh–Gordon formula, one finds that it has exactly two components isomorphic to $W^+ \cong V^+ \otimes L^{\frac{1}{2}}$. Now $\nabla_A \nabla_A \Phi \in T^* \otimes T^* \otimes V^+ \otimes L^{\frac{1}{2}}$. In this case, the two W^+ components must be given by $\nabla_A^* \nabla_A \Phi$ and $D_A^2 \Phi$.*

On the other hand, the Ricci identity tells us that the components of $\nabla_A\nabla_A\Phi$ in $\bigwedge^2 \otimes W^+$ must be determined by the curvature of ∇_A applied to Φ. Since the Spin^c *connection is determined from the Levi–Civita connection and the connection on L, its curvature is determined by the Riemann curvature tensor and F_A. But* $\mathrm{End}(W^+) \cong S^2V^+ \oplus \mathbb{C}$. *Thus the only pieces of the curvature that can act on Φ to give an element of W^+ are the scalar curvature and F_A^+.*

Thus there must be some linear relation between $\nabla_A^*\nabla_A\Phi$, $D_A^2\Phi$, $F_A^+\Phi$ and $s\Phi$. A check in a particular case suffices to find the constants. $\square$

2.3.2 The Seiberg–Witten invariants

We are now in a position to give a synopsis of the Seiberg–Witten invariants. To save time we shall only discuss the mod 2 Seiberg–Witten invariants — this allows us to evade the issue of orienting the moduli space. Also we only discuss the case when the virtual dimension of the moduli space is zero. This allows us to avoid discussing the topology of the parameter space. Thus it is possible to define far more general invariants than the ones we discuss. Also we shall only attempt to give the basic ideas behind the definitions. We shall ignore all the analytic details required to back up our definition. Any reader interested in these details could look in [Mor96] where a careful definition of the invariants is given.

The starting point is to take an oriented Riemannian 4-manifold (M, g) together with a Spin^c structure c. One considers the *Seiberg–Witten* equations for a section Φ of W^+ and a connection A on L:

$$F_A^+ = (\Phi \otimes \overline{\Phi})_0,$$
$$D_A\Phi = 0.$$

$\overline{\Phi} \in V^+ \otimes L^{-\frac{1}{2}}$ is determined by $\tilde{}$ on the V^+ factor and complex conjugation on the $L^{\frac{1}{2}}$ factor of W^+. Thus $(\Phi \otimes \overline{\Phi})_0$ is a trace free, pure imaginary, endomorphism of W^+ and hence lies in $i[S^2V^+]$ as does F_A^+.

Of course given one solution (A, Φ) to the Seiberg–Witten equations, one can always generate another by means of a *gauge transformation*. That is one chooses a smooth section $e^{i\theta}$ of S^1 and uses it to rotate the S^1 factor of the *Spin*^{c} structure. This has the effect of adding $i d\theta$ to the connection A and multiplying the spinor Φ by $e^{i\theta}$. Thus to any solution of the equations, there is an infinite family of gauge equivalent solutions. We shall be interested in the *moduli space* $\mathcal{M}$ which consists of all gauge equivalence classes of solutions to the Seiberg–Witten equations.

One might hope that $\mathcal{M}$ would be a finite-dimensional smooth manifold. A starting point to showing this is, of course, a deformation complex. Infinitesimal changes of gauge lie in $\Gamma(i\mathbb{R})$. Infinitesimal changes of connection lie in $\Gamma(T^* \otimes i\mathbb{R})$. Infinitesimal changes in a spinor or in the curvature of L lie in $\Gamma(W^{\pm})$ and $\Gamma(i\bigwedge^+ \oplus i\bigwedge^-)$. With these identifications in mind we have the deformation complex:

$$0 \longrightarrow \Gamma(i\mathbb{R}) \xrightarrow{d_1} \Gamma(W^+ \oplus T^* \otimes i\mathbb{R}) \xrightarrow{d_2} \Gamma(i\bigwedge^+ \oplus W^-) \longrightarrow 0$$

where d_1 is the linearisation of the action of a gauge transformation and d_2 is the linearisation of the Seiberg–Witten equations. Around a solution (A, Φ) to the Seiberg–Witten equations, the above complex is an elliptic complex with index $d = \frac{1}{4}(c_1(L)^2 - 2\chi - 3\tau)$. To see this one need only observe that, ignoring 0-th order operators, the Seiberg–Witten equations are just the self-duality equation and the Dirac equation. One can now appeal to the corresponding results for these equations.

Standard theory suggests, and it can be proved, that so long as d_2 is surjective at our solution (A, Φ) and, so long as the action of the gauge group on (A, Φ) has no stabiliser, then, near (A, Φ) $\mathcal{M}$ will be a smooth manifold of dimension d.

Thus there are only two reasons why $\mathcal{M}$ might fail to be a smooth manifold.

To deal with the problem that we need d_2 to be surjective, one introduces the *perturbed Seiberg–Witten equations*:

$$\begin{aligned} F_A^+ &= (\Phi \otimes \overline{\Phi})_0 + \eta \\ D_A \Phi &= 0 \end{aligned}$$

where η is a section of $i\bigwedge^+$. It can be proved, using the Sard–Smale theorem, that for generic η, d_2 will be surjective.

In considering the second problem — namely that the gauge group must not have any stabiliser — one first notices that if the gauge group has non trivial stabiliser one must have that $\Phi \equiv 0$. In this case the solution is called *reducible*. The perturbed Seiberg–Witten equations reduce to simply $F_A^+ = \eta$. By standard Hodge theory, there exists such a connection A on L

if and only if the projection of η onto the space of self-dual harmonic forms is equal to the projection of $2\pi c_1(L)$ onto the space of self-dual harmonic forms.

Thus the space of η for which no reducible solutions exist is of codimension b^+ in the space of all self-dual forms $\eta \in i\bigwedge^+$.

We conclude that on a manifold with $b^+ > 0$, if one chooses η generically then $\mathcal{M}$ will be a smooth manifold of dimension $d = \frac{1}{4}(c_1(L)^2 - 2\chi - 3\tau)$. Crucially, $\mathcal{M}$ is also compact. The essential ingredient in the proof of this is the following C^0 estimate on solutions of the equations:

Proposition 2.3.2 *If (A, Φ) is a non-reducible solution of the unperturbed Seiberg–Witten equations then $2\sqrt{2}|F_A^+| = |\Phi|^2 \leq \sup(-s)$.*

Proof: *At a maximum of $|\Phi|^2$, one has $\Delta|\Phi|^2 \geq 0$, so*

$$\begin{aligned} 0 \leq \Delta|\Phi|^2 &= 2(\nabla^*\nabla\Phi, \Phi) - 2|\nabla\Phi|^2 \\ &= -\frac{s}{2}|\Phi|^2 - \frac{1}{2}|\Phi|^4 - 2|\nabla\Phi|^2 \end{aligned}$$

by the Weitzenböck formula and the fact that (A, Φ) solve the equations. So $|\Phi|^2 \leq -s$ at the maximum with equality if and only if $\nabla\Phi = 0$. □

Incidentally, notice that it follows that the scalar curvature must be somewhere negative if the Seiberg–Witten equations are to have any solutions which are not reducible.

Thus if one has a Spin^c structure, c, on a compact, oriented Riemannian 4-manifold (M, g) with $b^+ > 0$, and with $d = 0$ one can attempt to define the *Seiberg–Witten* invariant of (M, c, g, η) to be the number of solutions, counted mod 2 to the perturbed Seiberg–Witten equations. Of course, the hope is that, as one smoothly varies the metric and η, one obtains a compact cobordism between the different moduli spaces. If one can prove that this is the case then the Seiberg–Witten invariant will depend only on M and the homotopy class of c.

Much the same arguments as those required to prove that for generic η, $\mathcal{M}$ is a compact, smooth manifold allow one to show that one does indeed obtain a compact cobordism if one varies g and η along a generic smooth path in the space $\mathcal{P}$ of possible metrics and perturbations so long as one never passes through the *wall* $W \subseteq \mathcal{P}$ of (g, η) for which reducible solutions exist. If $b^+ > 1$ then the wall is of codimension greater than 1 and so a generic path avoids the wall. Thus on a 4-manifold with $b^+ > 1$ the Seiberg–Witten invariant depends only on the diffeomorphism type of the manifold and the homotopy class of c.

If $b^+ > 1$, the wall is of codimension 1 and splits the parameter space into two components $\mathcal{P}^+$ and $\mathcal{P}^-$. The so called *wall-crossing formula* tells us what happens to the Seiberg–Witten as one crosses from one side of the wall to the other. For example if the manifold is simply connected then the invariant changes parity, [KM94]. For the general wall crossing formula, see [OO96a] or [LL95].

For definiteness, one distinguishes $\mathcal{P}^+$ and $\mathcal{P}^-$ by picking an orientation, σ, for $H^2_+(M^4, \mathbb{R})$. Then for any metric g, there exists a unique harmonic form ω^+ of unit norm which is positively oriented with respect to σ. Then (g, η) lies in $\mathcal{P}^\pm$ according to the sign of:

$$\int \omega^+ \wedge (c_1 - \eta).$$

With this convention one can define the Seiberg–Witten invariants, denoted $SW^\pm(M, c, \sigma)$ of a compact oriented 4-manifold with $b^+ = 1$ and a Spin^c structure c and orientation σ for $H^2_+(M, \mathbb{R})$ to be the Seiberg–Witten invariant calculated using a generic $(g, \eta) \in \mathcal{P}^\pm$.

2.3.3 The Seiberg–Witten invariants on symplectic manifolds

Suppose that (M^4, g, J) is an almost-Hermitian manifold. It is easy to see that we have a canonical Spin^c structure. As usual we locally choose a Spin structure with associated bundles $V^\pm$. Since our manifold is almost-Hermitian, we have a spinor representative u for J and hence the line bundle $L^{-\frac{1}{2}}$ defined by $\langle \tilde{u} \rangle$. Although the bundles $L^{-\frac{1}{2}}$ and $V^\pm$ are defined only up to a choice of Spin structure, the associated Spin^c structure with bundles

$$W^+ \cong V^+ \otimes L^{\frac{1}{2}} \cong \mathbb{C} \oplus \bigwedge\nolimits^{0,2},$$

$$W^- \cong V^- \otimes L^{\frac{1}{2}} \cong \bigwedge\nolimits^{0,1},$$

$$L \cong \bigwedge^{0,2},$$

is well defined.

Notice that $c_1(L)^2 = 2\chi + 3\tau$. Thus the virtual dimension d of the moduli space of solutions to the Seiberg–Witten equations for this Spin^c structure is 0, and hence so long as $b^+ \geq 1$ the Seiberg–Witten equations will be defined.

Suppose now that M is almost-Kähler. In this circumstance, Taubes' considered the perturbed equations:

$$\begin{aligned} F_A^+ &= \Phi \otimes \overline{\Phi} + F_{A_0}^+ - ir\omega \\ D_A\Phi &= 0 \end{aligned}$$

where A_0 is the connection on $\bigwedge^{0,2} = L$ determined by $\overline{\nabla}$ and $r \in R$ is some constant.

In [Tau94], Taubes proved that for sufficiently large r, the perturbed Seiberg–Witten invariant is 1. Thus in the case where $b^+ > 1$, the Seiberg–Witten invariant of the canonical Spin^c structure on an almost-Kähler manifold is equal to 1. In the case when $b^+ = 1$, we can use ω to define an orientation of $H^2_+(M, \mathbb{R})$ since it is self dual and harmonic. When r is self-dual and harmonic, the sign of

$$\int \omega \wedge (c_1 - F_{A_0}^+ + ir\omega)$$

is, of course, positive. In conclusion we have:

Theorem 2.3.3 *[Tau94] If M^4 is a compact symplectic manifold then if $b^+ > 1$ the Seiberg–Witten invariant of the canonical* Spin^c *structure is* 1*. If $b^+ = 1$ then SW^+ of the canonical* Spin^c *structure is* 1*.*

This powerful result is in fact only the starting point of Taubes' deep investigations into the relationships between symplectic topology and the Seiberg–Witten invariants. He has gone on to prove the equivalence of the perturbed Seiberg–Witten invariants and certain Gromov invariants ([Tau95a], [Tau95b], [Tau96]). The Gromov invariants of a symplectic manifold are prima-facie only invariants of the (deformation class) of the symplectic structure, so this result is extremely striking. Indeed, once one combines Taubes' results with theorems' of Gromov's and McDuff's one is able to prove some important uniqueness results on the existence of symplectic structures on given manifolds — see [LM96].

Nevertheless, this result suffices to illustrate the importance of the Seiberg–Witten invariants to questions of symplectic topology. Also, it has an immediate application to our study of almost-Kähler Einstein manifolds.

Theorem 2.3.4 *[OO96b] If M^4 admits a metric of everywhere positive scalar curvature and satisfies $2\chi + 3\tau \geq 0$ then any symplectic form ω on M satisfies $c_1(M) \wedge [\omega] \leq 0$.*

Proof: *Let g_1 be an almost-Kähler metric compatible with ω and let g_0 be a metric of everywhere positive scalar curvature. Hence the unperturbed Seiberg–Witten invariants vanish. Hence by Taubes' theorem, $b^+ = 1$. Now $c_1(\bigwedge^{0,2})^2 = 2\chi + 3\tau \geq 0$ and so it is not possible for $c_1(\bigwedge^{0,2})$ to be represented by an anti-self-dual form with respect to any metric. Thus if one varies the metric from g_0 to g_1 but keeps the perturbation as zero, wall-crossing cannot occur. Thus the unperturbed Seiberg–Witten invariant for the metric g_1 and the canonical* Spin^c *structure must be zero. Thus g_0 must lie in $\mathcal{P}^-$ and hence:*

$$\int c_1(\bigwedge^{0,2}) \wedge \omega \geq 0.$$

This gives the desired result. $\square$

Corollary 2.3.5 *If M^4 is compact and admits a metric of everywhere positive scalar curvature then it cannot admit a strictly almost-Kähler, Einstein structure.*

Proof: *If M admits an Einstein metric we must have $2\chi + 3\tau \geq 0$ by the Hitchin–Thorpe inequality. The result now follows from Proposition 2.2.11* $\square$

The manifolds $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $0 \leq k \leq 8$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ all admit both symplectic structures (clearly) and metrics of everywhere positive scalar curvature ([Hit75]). Thus they cannot admit strictly almost-Kähler Einstein metrics. Moreover, $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ obviously admit Einstein metrics. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ admits the Page metric which is Einstein [Pag79]. Finally, it was proved in [Tia87] that $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ all admit Kähler Einstein metrics of positive scalar curvature when $3 \leq k \leq 8$.

We thus obtain examples of manifolds which are known to admit both symplectic structures and Einstein metrics but which cannot admit strictly almost-Kähler structures. To the author's knowledge these are the only known examples apart from the rather trivial cases of manifolds for which the moduli space of Einstein metrics is completely known. Our result cannot be used to generate other examples since it is proved in [OO97] that smooth 4 manifolds which admit a metric of positive scalar curvature and a symplectic structure are diffeomorphic to rational or ruled surfaces — this is one of the powerful results obtained by combining Taubes' results with previously known results on Gromov invariants. Irrational ruled surfaces all have non-positive Euler characteristic and so cannot admit Einstein metrics.

2.3.4 The Miyaoka–Yau inequality

The C^0 estimate on solutions of the Seiberg–Witten equations provides a basic link between the Seiberg–Witten invariants and Riemannian geometry. The essential point is that, given a Spin^c structure with non-vanishing invariants, we obtain a lower bound on the scalar curvature in terms of F_A^+ and thence from the Chern class of the line bundle L. Thus we gain estimates on the scalar curvature from information on the differential topology. This was observed by Witten in [Wit94] and has been used to good effect by LeBrun in a number of papers ([LeB95a], [LeB96], [LeB95b] for example). We describe the first of his results now:

Theorem 2.3.6 *[LeB95a] If M^4 is a compact Einstein manifold with non-vanishing unperturbed Seiberg–Witten invariant then $\chi \geq 3\tau$ with equality if and only if M is a finite quotient of T^4 or $\mathbb{C}H^2$.*

Remark: Since M is Einstein, $c_1(L)^2 = 2\chi + 3\tau \geq 0$ and so wall crossing does not occur as one varies the metric. Thus one does not need any proviso about the size of b^+.

Proof: *We have a solution (A, Φ) to the unperturbed Seiberg–Witten equations.*

$$
\begin{aligned}
2\chi + 3\tau = c_1(L)^2 &= \frac{1}{4\pi^2}\int (|F_A^+|^2 - |F_A^-|^2) \\
&\leq \frac{1}{4\pi^2}\int |F_A^+|^2 \\
&\leq \frac{1}{32\pi^2}\int s^2.
\end{aligned}
$$

We have equality if and only if $F_A^- = 0$ and $\nabla\Phi = 0$. Since $F_A^+ = \Phi \odot \tilde{(\Phi)}$, we also have that $\nabla F_A^+ = 0$ when we have equality, so F_A^+ describes a Kähler structure compatible with the metric.

Using the integral formulae for χ and τ from the proof of the Hitchin–Thorpe inequality, one has

$$
\frac{1}{32\pi^2}\int s^2 \leq \left[\frac{1}{4\pi^2}\int\left(\frac{s^2}{24} + \|W^-\|^2\right)\right] = 3(2\chi - 3\tau)
$$

with equality if and only if the manifold is self-dual.

Combining these two inequalities on has that $\chi \geq 3\tau$ with equality if and only if the manifold is Einstein, Kähler and self-dual. Thus in the case of equality one has that $\nabla R = 0$, so the manifold is symmetric. It follows immediately that M is isometric to a finite quotient of $\mathbb{C}P^2$, T^4 or $\mathbb{C}H^2$. But $\mathbb{C}P^2$ has positive scalar curvature and so cannot possibly have non-vanishing unperturbed Seiberg–Witten invariant. □

Corollary 2.3.7 *If M^4 is compact and*

- *admits a symplectic structure*
- *admits an Einstein metric*

then M satisfies $\chi \geq 3\tau$

Proof: *If $b^+ > 1$, one merely has to combine Taubes' and LeBrun's results. In the case where $b^+ = 1$, the inequality is implied by the Hitchin–Thorpe inequality.* □

Corollary 2.3.8 *If M is diffeomorphic to a compact quotient of $\mathbb{C}H^2$ then it does not admit a strictly almost-Kähler structure with respect to either orientation.*

Proof: Using Hodge theory one can easily prove that on a compact Kähler 4 manifold any almost-Kähler structure compatible with the metric and the orientation must be Kähler. On the other hand Theorem 2.2.4 tells us that a compact, Einstein, self-dual manifold cannot admit a strictly almost-Kähler structure compatible with the metric and the opposite orientation. $\square$

Notice that the uniqueness of the almost-Kähler Einstein metric on $\mathbb{C}P^2$ does not follow from LeBrun's theorem.

In fact LeBrun has refined this result to find further topological obstructions to the existence of Einstein metrics on smooth four manifolds — see [LeB96]. The essential idea in this paper is that if one has more than one non-vanishing Seiberg–Witten invariant, for example on a blow-up of a symplectic manifold, then one can obtain more restrictive bounds on the scalar curvature.

Chapter 3

Weakly $*$-Einstein manifolds

3.1 Introduction

In this chapter we shall prove the non-existence of almost-Kähler structures under various curvature assumptions.

We are motivated in particular by the problem of proving that four dimensional hyperbolic space does not admit an almost-Kähler structure even locally — a problem which has, surprisingly, remained unsolved for some time. A proof that hyperbolic space of dimension 8 or above does not admit a compatible almost-Kähler structure was given in [Ols78]. Unfortunately, the proof of the general case given in [Ogu97] contains a gap, as does the proof in the 4 dimensional case given in [Bla90].

This problem raises the more general question of "How can one tell if a given Riemannian metric locally admits a compatible almost-Kähler structure?" We shall develop a strategy which one can in principle apply to answer this question with the aim of applying it to the particular case of hyperbolic space. However, it becomes clear that one can easily adapt the proof to show that:

Theorem 3.1.1 *If (M^4, g) is an anti-self-dual, Einstein, Riemannian manifold than it cannot admit a strictly almost-Kähler structure, even locally.*

and this is the result we shall prove to illustrate our strategy.

With the increased understanding of almost-Kähler geometry that we acquire during the course of the above proof we shall in fact be able to go rather further and prove:

Theorem 3.1.2 *All strictly almost-Kähler, Einstein, weakly $*$-Einstein 4-manifolds are described by Tod's construction.*

Because our eventual aim in this chapter is to prove this result, we shall work throughout with quite a high degree of generality. An unfortunate consequence is that the original motivation of considering hyperbolic space may at times be somewhat obscured. Also, the formulae will become rather lengthy. The reader may find it helpful to bear in mind the example of hyperbolic space — for in that case many of our formulae become trivial, and the importance of the strategy in proving the results becomes very clear.

Having proved theorem 3.1.2, we show how our ideas can be used in dimensions 6 and above. Specifically we shall prove:

Theorem 3.1.3 *Constant curvature manifolds of any dimension which admit an almost-Kähler structure are necessarily flat and Kähler*

3.2 The strategy

Suppose that (M^4, g) is a Riemannian manifold. To find out if it admits an almost-Kähler structure, we must solve the equation $\mathrm{d}\omega = 0$ for almost-complex structures J compatible with the metric. The natural thing to do therefore is to apply Cartan–Kähler theory. Rather than do this explicitly, we shall indicate how to reduce the problem to the Frobenius theorem.

Suppose we have a 1-jet solution $(J, \nabla J)$ (equivalently (J, ξ)) to the problem. Now consider the Ricci identity:

$$2\alpha\nabla\nabla J = RJ. \tag{3.1}$$

We analysed this equation in Theorem 1.2.2, and saw that three of the U(2) components of $\overline{\nabla}\xi$ and the norm of ξ are determined by the curvature R. Thus the condition $\|\xi\|^2 = -b$ is an obstruction to lifting a 1-jet solution of the problem to a 2-jet solution.

Since $\|\xi\|^2 = -b$, $2(\overline{\nabla}\xi, \xi) = -\mathrm{d}b$. Combining this with the fact that three components of $\overline{\nabla}\xi$ are determined by R, one can show that $\nabla\xi$ is determined by J, ξ, R, $\overline{\nabla}R$ up to a two real-dimensional parameter which we shall call γ. So a 2-jet solution can be thought of as a triple (J, ξ, γ).

Now consider the Ricci identity:

$$2\alpha\nabla\nabla\xi = R\xi. \tag{3.2}$$

Dimension counting indicates that this equation will (a) give some restrictions on (J, ξ, γ) — i.e. an obstruction to lifting 2-jet solutions to 3-jet solutions and (b) determine $\nabla\gamma$.

So we have a parameter space P consisting of those (J, ξ, γ) satisfying the condition (a). Moreover, given such a triple $\Phi = (J, \xi, \gamma)$ we can compute what $\nabla\Phi$ must be in terms of R and its derivatives. Thus we shall get an equation of the form $\nabla\Phi = f(\Phi)$ which can be tackled by applying the Frobenius theorem. It is not too hard to see that the Ricci-identity:

$$2\alpha(\nabla\nabla\gamma) = R\gamma \tag{3.3}$$

is the resulting integrability condition.

Thus to find out if a given Riemannian metric on a 4-manifold locally admits an almost-Kähler structure, the strategy one should employ is to find out whether or not there exist algebraic examples of 3-jet solutions satisfying equations (3.1), (3.2) and (3.3).

In summary then, our method is to attempt to construct a 1-jet, then a 2-jet, then a 3-jet solution to the problem, making sure at each stage that our solution satisfies the requirement that derivatives commute — equivalently that it satisfies the Ricci-identity. We continue in this way until we show that it is algebraically impossible to find an n-jet solution, or until we find that we can apply some general existence theorem — in this case, Frobenius' theorem. This is precisely the strategy used in Cartan–Kähler theory with the small modification that we appeal to Frobenius' theorem rather than the Cartan–Kähler theorem at the end of the process.

Note that the technique for finding out if a given Riemannian metric admits a compatible almost-Kähler structure is substantially more complex than the comparable method for finding if one admits a compatible Hermitian structure (see Section 1.2.2).

In dimensions greater than 4, similar strategies can be devised. In some regards they are easier as one need not consider as many derivatives — though the algebra becomes compensatingly more messy. We shall use this in Section 3.5 to prove that real hyperbolic 6-space cannot admit an almost-Kähler structure. Also, we shall show how to recast the proof given in [Ols78] that hyperbolic space of dimension 8 or above cannot admit an almost-Kähler structure so that it conforms to our strategy.

3.3 Applying the strategy

3.3.1 Analysing equation (3.1)

Equation (3.1) has, of course, been considered already — but we wish to re-examine it in spinor notation. Thus we shall actually consider the equivalent equation $2\alpha\nabla\nabla u = Ru$.

Since M is almost-Kähler we have that $\nabla u = \phi \otimes u + \beta v \otimes \tilde{u} \otimes \tilde{u}$ where $u \in V^+$ represents the almost-complex structure.

The first step is to write the curvature tensor R of g in spinor notation. We can view R as lying in $(S^2V^+ \oplus S^2V^-) \otimes (S^2V^+ \oplus S^2V^-)$. This allows us to write:

$$\begin{aligned} R \;=\; & \left(\frac{s^*}{4}u\tilde{u} + w_F^+ u^2 - \overline{w_F^+}\tilde{u}^2 + r_f\right) \otimes u\tilde{u} \\ & + 2\Re\left(\left(w_F^+ u\tilde{u} + w_{00}^+ u^2 + \frac{s^*-s}{16}\tilde{u}^2 + r_{00}\right) \otimes u^2\right) \\ & + u\tilde{u} \otimes r_f + u^2 \otimes r_{00} + \left(W^- + \frac{s}{12}\mathbf{1}\right) \end{aligned} \tag{3.4}$$

with w_F^+, $w_{00}^+ \in \mathbb{C}$, $r_f \in [S^2V^-]$, $r_{00} \in S^2V^-$. Most of the above equation can be viewed as a definition of the terms within it — the only content is ensuring that the coefficients of s^* and $s^* - s$ are correct. Of course, we see that w_F^+ corresponds to W_F^+ and so on. (Incidentally, by our comments on the representation theory of U(2) and spinors, we have just proved that our decomposition of R is indeed into irreducibles.)

Let us now define spinors P, $Q \in S^2V^-$ and p, $q \in \mathbb{C}$ by

$$\begin{aligned}\overline{\nabla}\beta\hat{v} = \overline{\nabla}(\beta v \otimes \tilde{u} \otimes \tilde{u}^2) &= (P+p) \otimes u \otimes \tilde{u} \otimes \tilde{u}^2 \\ &\quad + (Q+q) \otimes \tilde{u} \otimes \tilde{u} \otimes \tilde{u}^2,\end{aligned} \tag{3.5}$$

where we think of $P+p$ and $Q+q$ as lying in $V^- \otimes V^-$. Here, P, p are both of weight -2, and Q, q are both of weight -4.

Lemma 3.3.1 *If M is an almost-Kähler 4-manifold then we have,*

$$p = \overline{w_F^+}, \quad q = -\overline{w_{00}^+}, \quad P = -\overline{r_{00}}, \quad \|\beta\|^2 = \frac{s^* - s}{16},$$

$$\mathrm{d}\phi = (\beta^2 - \frac{s^*}{8})u\tilde{u} + \frac{w_F^+}{2}u^2 - \frac{\overline{w_F^+}}{2}\tilde{u}^2 - \frac{r_f}{2} - \beta^2 v\tilde{v}.$$

Proof: The above equations are all gauge independent, because the terms on each side have the same weights. Thus it is sufficient to prove the result in a gauge where ϕ is zero at the point x. In such a gauge, we have that $\overline{\nabla}(\beta v \otimes \tilde{u} \otimes \tilde{u}^2) = (\nabla\beta v)\tilde{u} \otimes \tilde{u}^2$. So $\nabla\beta = (P+p) \otimes u + (Q+q) \otimes \tilde{u}$. Also we have:

$$\begin{aligned}\nabla\nabla u &= \nabla(\phi \otimes u + \beta v \otimes \tilde{u} \otimes \tilde{u}) \\ &= (\nabla\phi) \otimes u + (\nabla\beta v) \otimes \tilde{u} \otimes \tilde{u} \\ &\quad + \beta^2 \tilde{v} \otimes u \otimes v \otimes u \otimes \tilde{u} + \beta^2 \tilde{v} \otimes u \otimes v \otimes \tilde{u} \otimes u,\end{aligned}$$

using the fact that $\phi = 0$ at x to get from the first to the second line. Anti-symmetrising and using our formula for $\nabla(\beta v)$ we now have:

$$\begin{aligned}\alpha(\nabla\nabla u) &= (\mathrm{d}\phi) \otimes u + pu\tilde{u} \otimes \tilde{u} + P \otimes \tilde{u} + q\tilde{u}^2 \otimes \tilde{u} \\ &\quad - \beta^2 u^2 \otimes \tilde{u} - \beta^2 u\tilde{u} \otimes u + \beta^2 v\tilde{v} \otimes u,\end{aligned}$$

On the other hand we have $2\alpha\nabla\nabla u = Ru$, hence we can write down the curvature tensor of V^+:

$$\begin{aligned}R^{V^+} &= (-2\mathrm{d}\phi + 2\beta^2 u\tilde{u} - 2\beta^2 v\tilde{v}) \otimes u\tilde{u} \\ &\quad + 2\Re((-2pu\tilde{u} + 2\beta^2 u^2 - 2q\tilde{u}^2 - 2P) \otimes \tilde{u}^2).\end{aligned}$$

Thus the result follows by comparing this with the formula for the curvature tensor of $TM \cong V^+ \otimes V^-$ that we wrote down before. $\square$

By way of an example, if (M^4, g) is hyperbolic space, we can choose an algebraic example of a 2-jet solution $(\omega, \xi, \overline{\nabla}\xi)$ to the equation $\mathrm{d}\omega = 0$ by choosing any unit $u \in V^+$, any unit $v \in V^-$, $\beta \in \mathbb{R}$ with $2\beta^2 = *s$, and any $Q \in S^2V^-$. One then insists that $\overline{\nabla}(\beta v) = Q \otimes \tilde{u} \otimes \tilde{u} \otimes \tilde{u}^2$. From u one can compute ω, from v and β one can compute ξ and from $\overline{\nabla}(\beta v)$ one can compute $\overline{\nabla}\xi$. Thus we have completely identified the space of 2-jet solutions to the problem of finding an almost-Kähler structure on hyperbolic space. In particular, there are solutions! So one must examine higher derivatives if one is to have any hope of proving that hyperbolic space cannot locally admit an almost-Kähler structure.

3.3.2 An application of the differential Bianchi identity

We wish to apply our strategy to more than just hyperbolic space. As our strategy involves considering higher derivatives of the curvature, we shall be forced to consider such matters as the differential Bianchi identity. Of course, this is entirely trivial in the case of hyperbolic space since $\nabla R = 0$ on hyperbolic space. Thus this section is really a digression from the strategy.

As observed in [Sal82], Spinors provide a quick way to get to grips with the differential Bianchi identity on 4-manifolds. Since $W^+ \in S^4V^+$,

$$\nabla W^+ \in S^5V^+ \otimes V^- \oplus S^3V^+ \otimes V^-.$$

Similarly,

$$\nabla R_0 \in S^3V^+ \otimes S^3V^- \oplus S^3V^+ \otimes V^- \oplus V^+ \otimes S^3V^- \oplus V^+ \otimes V^-$$

and $\mathrm{d}s \in V^+ \otimes V^-$.

On the other hand if we define

$$B : \bigwedge^1 \otimes \mathcal{R} \subseteq \bigwedge^1 \otimes \bigwedge^2 \otimes \bigwedge^2 \longrightarrow \bigwedge^3 \otimes \bigwedge^2$$

by antisymmetrisation on the first three factors, the differential Bianchi identity reads $B(\nabla R) = 0$. So since

$$\bigwedge^3 \otimes \bigwedge^2 \cong S^3V^+ \otimes V^- \oplus 2V^+ \otimes V^- \oplus V^+ \otimes S^3V^-,$$

we shall have, by Schur's lemma, that the components of ∇W^+ and ∇R_0 in $S^3V^+ \otimes V^-$ are essentially equal. Thus on an Einstein 4-manifold, the component of ∇W^+ in $S^3V^+ \otimes V^-$ is zero.

Let us suppose then that (M^4, g, J) is an almost-Kähler, Einstein 4-manifold and that u is a spinor representative of J as before. Let us write

$$\phi = \Phi \otimes u - \tilde{\Phi} \otimes \tilde{u} \tag{3.6}$$
$$\mathrm{d}w^+_{00} = A \otimes u + B \otimes \tilde{u} \tag{3.7}$$
$$\mathrm{d}s^* = \alpha \otimes u \tilde{+} \alpha \otimes \tilde{u} \tag{3.8}$$

and we shall assume in this section that $w^+_F = 0$ — i.e. the manifold is weakly $*$-Einstein. From our formula (3.4) for R we can write W^+ as:

$$W^+ = w^+_{00} u^4 + \frac{(3s^* - s)}{8} u^2 \tilde{u}^2 + \overline{w^+_{00}} \tilde{u}^4. \tag{3.9}$$

We can compute ∇W^+ from this:

$$\begin{aligned} \nabla W^+ \quad = \quad & A \otimes u \otimes u^4 + B \otimes \tilde{u} \otimes u^4 \\ & + 4w^+_{00} \phi \otimes u^4 + 4w^+_{00} \beta v \otimes \tilde{u} \otimes u^3 \otimes \tilde{u} \\ & + \frac{3}{8} \alpha \otimes u \otimes u^2 \otimes \tilde{u}^2 + \frac{2(3s^* - s)}{8} \beta \tilde{v} \otimes u \otimes u^3 \tilde{u} \\ & + \text{conjugate.} \end{aligned}$$

We can now take the component of this in $S^3V^+ \otimes V^-$:

$$\begin{aligned} (\nabla W^+)^{S^3V^+ \otimes V^-} \quad = \quad & (-B + 4w^+_{00} \tilde{\Phi} + \frac{3s^* - s}{16} \beta \tilde{v}) u^3 \\ & + (-3w^+_{00} \beta v + \frac{3}{16} \alpha) u^2 \tilde{u} \\ & + \text{conjugate.} \end{aligned}$$

We deduce:

Proposition 3.3.2 *On an almost-Kähler, Einstein, weakly $*$-Einstein 4-manifold:*

$$B = 4w^+_{00} \tilde{\Phi} + \frac{(3s^* - s)}{16} \beta \tilde{v} \tag{3.10}$$
$$\alpha = 16 w^+_{00} \beta v. \tag{3.11}$$

Equivalently by (3.7) and (3.8),

$$\mathrm{d}s^* = 16w_{00}^{+}\beta v \otimes u + 16\overline{w_{00}^{+}}\beta\tilde{v} \otimes \tilde{u}, \tag{3.12}$$

$$\mathrm{d}w_{00}^{+} = A \otimes u + \frac{(3s^* - s)}{16}\beta\tilde{v} \otimes \tilde{u} + 4w_{00}^{+}\tilde{\Phi} \otimes \tilde{u}. \tag{3.13}$$

As a consequence we have:

Corollary 3.3.3 *Suppose M is strictly almost-Kähler, Einstein, weakly $*$-Einstein and either s^* is constant or $w_{00}^{+} \equiv 0$ then M will in fact be ASD.*

Proof: By equation (3.12), if s^* is constant then we must have $w_{00}^{+} \equiv 0$. The fact $w_{00}^{+} \equiv 0$ implies that $\mathrm{d}w_{00}^{+} \equiv 0$ so taking the $\tilde{v} \otimes \tilde{u}$ term of equation (3.13) one sees that $s^* = \frac{s}{3}$. Comparing this with our formula (3.9) for W^+ we see that $W^+ \equiv 0$. $\square$

3.3.3 ASD, Einstein, strictly almost-Kähler manifolds

We return to the strategy now and complete our analysis of equation (3.1). Suppose that M^4 is strictly almost-Kähler, Einstein and ASD.

By Lemma 3.3.1 and (3.5) we see that:

$$\overline{\nabla}(\beta v \otimes \tilde{u} \otimes \tilde{u}^2) = Q \otimes \tilde{u} \otimes \tilde{u} \otimes \tilde{u}^2 \tag{3.14}$$

where Q is in S^2V^-. Hence we have

$$\nabla\beta v = Q \otimes \tilde{u} - 3\overline{\phi} \otimes \beta v = Q \otimes u + 3\phi \otimes \beta v$$

Again by Lemma 3.3.1 we have that $\beta^2 = \frac{s^*-s}{32} = -\frac{s}{48}$. This tells us that $s \leq 0$ and that M is automatically Kähler if $s = 0$ and never Kähler if $s < 0$. Thus we assume from now on that $s < 0$.

Since s is constant, $\mathrm{d}\beta^2 = 0$. Equivalently,

$$\eta_-(\nabla\beta v, \beta\tilde{v}) + \eta_-(\beta v, \nabla\beta\tilde{v}) = 0.$$

If we write $Q = q^1\beta^2v^2 + q^2\beta^2v\tilde{v} + q^3\beta^2\tilde{v}^2$ then we must have, by equation (3.14):

$$q^1\beta^3 v \otimes \tilde{u} + q^2\beta^3\tilde{v} \otimes \tilde{u} + \text{conjugate} = 0.$$

Thus $q^1 = q^2 = 0$. We can now define the complex function γ mentioned in the strategy, it is given by $\gamma = q^3$. Note that γ has weight -2.

This completes the analysis of equation (3.1) in the ASD, Einstein case. We are now ready to begin the analysis of equation (3.2). We shall in fact work with the equivalent equation $2\alpha\nabla\nabla(\beta v) = R(\beta v)$.

So equation (3.14) now reads:

$$\nabla\beta v = \gamma\beta^2\tilde{v}\otimes\tilde{u}\otimes\tilde{v} + 3\beta\phi\otimes v. \tag{3.15}$$

We assume (without loss of generality) that $\phi = 0$ at x, so we can differentiate this to get

$$\begin{aligned}\nabla\nabla\beta v \;=\;& \beta^2(\mathrm{d}\gamma)^1\otimes u\otimes\tilde{v}\otimes\tilde{u}\otimes\tilde{v} + \beta^2(\mathrm{d}\gamma)^2\otimes\tilde{u}\otimes\tilde{v}\otimes\tilde{u}\otimes\tilde{v}\\ &- \beta^3|\gamma|^2 v\otimes u\otimes v\otimes\tilde{u}\otimes\tilde{v} - \beta^3|\gamma|^2 v\otimes u\otimes\tilde{v}\otimes\tilde{u}\otimes v\\ &+ \beta^3\gamma\tilde{v}\otimes u\otimes\tilde{v}\otimes u\otimes\tilde{v} + 3\beta\nabla\phi\otimes v\end{aligned}$$

where $\mathrm{d}\gamma = (\mathrm{d}\gamma)^1\otimes u + (\mathrm{d}\gamma)^2\otimes\tilde{u}$. We now anti-symmetrise and apply the formulae for $\mathrm{d}\phi$ and β^2 given in Lemma 3.3.1 to get:

$$\begin{aligned}\alpha\nabla\nabla\beta \;=\;& \frac{s}{48}\beta(|\gamma|^2 - 9)u\tilde{u}\otimes v\\ &+ \beta^2\eta_-((\mathrm{d}\gamma)^1,\tilde{v})u\tilde{u}\otimes\tilde{v} + \beta^2\eta_-((\mathrm{d}\gamma)^2,\tilde{v})\tilde{u}^2\otimes\tilde{v}\\ &+ \beta^3(-3 - |\gamma|^2)v\tilde{v}\otimes v - |\gamma|^2\beta^3 v^2\otimes\tilde{v} + \beta^2(\mathrm{d}\gamma)^1\tilde{v}\otimes\tilde{v}.\end{aligned}$$

As before, the formula $2\alpha\nabla\nabla\beta v = R\beta v$ allows us to write down the curvature of V^-. We can then compare this with our original formula for R. So the fact that M is Einstein implies that the terms on the first two lines of the R.H.S. are zero — i.e.

$$\|\gamma\|^2 = 9, \quad (\mathrm{d}\gamma)^1 = k^1\beta\tilde{v}, \quad (\mathrm{d}\gamma)^2 = k^2\beta\tilde{v}$$

where k^1, $k^2 \in \mathbb{C}$. So our formula simplifies to:

$$\alpha\nabla\nabla\beta v = -12\beta^3 v\tilde{v}\otimes v - 9\beta^3 v^2\otimes\tilde{v} + k^1\tilde{\beta}^3 v^2\otimes\tilde{v}.$$

So we have (assuming without loss of generality that $\beta = 1$):

$$R^{V^-} = 24v\tilde{v}\otimes v\tilde{v} + 18v^2\otimes\tilde{v}^2 + 18\tilde{v}^2\otimes v^2 - 2k^1\tilde{v}^2\otimes\tilde{v}^2 - 2\overline{k^1}v^2\otimes v^2.$$

To understand this formula, we note that $v\tilde{v} \in \bigwedge^-$ defines an almost-complex structure $\mathbb{J}$ compatible with g but with the *opposite* orientation.

This gives rise to a splitting of R as before, and in particular allows us to define W_{00}^-, W_F^- in the obvious way and $s^{*,-}$ to be the $*$-scalar curvature of this reverse oriented almost-Hermitian manifold. Our calculation of R^{V^-} can be compared with the formula 3.4 for R. This tells us that W_F^- vanishes, that $s^{*,-}$ is constant and that W_{00}^- is proportional to k^1. On the other hand, observe that equation (3.15) also tells us that $(M, g, \mathbb{J})$ is itself almost-Kähler! So we can apply Corollary 3.3.3 to see that (M, g) must be conformally flat. In summary then:

Lemma 3.3.4 *If (M, g, J) is an Einstein, ASD, almost-Kähler 4-manifold, then either $s = 0$ in which case M is scalar flat Kähler or else $s < 0$ and M is a constant curvature manifold. Moreover in this case we have that $|\gamma|^2$ is constant and also $(\mathrm{d}\gamma)^1 = 0$, $(\mathrm{d}\gamma)^2 = k^2\beta\tilde{v}$ at x whenever we use a gauge with $\phi = 0$ at x.*

This completes our analysis of equation (3.2). The analysis of equation (3.3) is very easy now. If (M, g, J) is a constant curvature, almost-Kähler 4-manifold then we have that when $\phi = 0$, $(\mathrm{d}\gamma)^1 = 0$ and $(\mathrm{d}\gamma)^2 = k^2\beta\tilde{v}$. Since $|\gamma|^2$ is constant, we have

$$(\mathrm{d}\gamma)\overline{\gamma} + \gamma(\mathrm{d}\overline{\gamma}) = 0.$$

Hence

$$\overline{\gamma}k^2\beta\tilde{v} \otimes \tilde{u} + \gamma\overline{k}^2\beta v \otimes u = 0.$$

So $\mathrm{d}\gamma = 0$ at x in any gauge where $\phi = 0$. We conclude that the tensor associated to γ, which is of weight -2, satisfies $\overline{\nabla}\hat{\gamma} \equiv 0$. Hence $\overline{R}\hat{\gamma} = 0$, where $\overline{R}$ is the curvature of $\overline{\nabla}$. It is easy to check that $\overline{R}\hat{\gamma}$ cannot equal zero because $\hat{\gamma}$ does not have weight zero. A contradiction.

In summary we have proved:

Theorem 3.3.5 *If (M^4, g, J) is an almost-Kähler 4-manifold which is both Einstein and weakly $*$-Einstein then if $W_{00}^+ = 0$, M must be Kähler.*

This, of course, includes the case of hyperbolic space and ASD manifolds.

3.4 Weakly $*$-Einstein manifolds

We can push this kind of analysis even further to prove that strictly almost-Kähler, Einstein, weakly $*$-Einstein manifolds are all given by Tod's construction. This result has the surprising corollaries that such manifolds are hyperkähler with regard to the opposite orientation and admit two commuting Killing vector fields. Moreover, the distribution $\mathcal{D}^{\perp}$ defined by the Nijenhuis tensor is spanned by commuting Killing vector fields and the opposite oriented almost-Kähler structure $\mathbb{J}$ defined by the Nijenhuis tensor is Kähler. It is in fact these observations which motivate the following proof. Since the proof is rather lengthy, we shall start by giving an overview of the central ideas.

Firstly we prove that $\mathbb{J}$ is Kähler by pushing the strategy we devised a little further. In the same way as we used the differential Bianchi identity to allow our analysis to apply to a wider class of problems, we shall now use the identity $\mathrm{d}^2 s^* = 0$. Once we have shown that $\mathbb{J}$ is Kähler it is not too surprising that $s = 0$. (Consider that J corresponds to u, $\mathbb{J}$ to v and that the Ricci form is determined, in large part, by $\mathrm{d}\phi$. Since u and v are gauge dependent with different weights we expect that this will force the scalar curvature to be zero).

Suppose that we are given a Riemannian 4-manifold and a two dimensional distribution. If we pick two vectors X, Y at a point in the distribution and attempt to extend these to two commuting vector fields which lie in the distribution, then dimension counting indicates that one can determine ∇X and ∇Y in terms of the values of X and Y at the point and the geometry of the distribution. Thus the question of whether or not *any* two vectors can be extended to satisfy the above conditions is of Frobenius type, and can be answered by checking the integrability condition. But note that if the distribution is spanned by two commuting Killing vector fields, then any vector in the distribution can be expressed as a linear combination of these Killing vector fields. So if one is given two vectors X, Y at a point, these can be extended to commuting Killing vector fields which lie in $\mathcal{D}$ if and only if *any* two vectors lying in $\mathcal{D}$ can be extended to commuting Killing vector fields. Hence the distribution is spanned by commuting Killing vector fields if and only if the Frobenius-type problem is integrable. In the case we are looking at it is easy to see that the integrability condition can only depend on the values of w^+_{00} and β. Since Tod's examples exhibit all possible values of w^+_{00} and β, we see that the integrability condition must always hold on an almost-Kähler, Einstein, weakly $*$-Einstein manifold.

The technical proof which follows is rather lengthy and contains a number of apparently miraculous cancellations. This is because the author has taken the trouble to use a consistent convention for working out coefficients when symmetrising and projecting tensors. One need not do this, the author's original proof found cancellations using Schur's lemma and the fact that all the results we prove are true for Tod's examples. The last two sentences of the previous paragraph give an example of the kind of argument one uses. Although the proof which avoids the need for carefully checking coefficients is somewhat more convincing, the author found that it was also almost entirely unreadable. Thus we shall work with careful conventions but shall explain the source of "miraculous" cancellations as they arise. One could view the careful proof as providing a double-check on the Schur's lemma type arguments.

3.4.1 $\mathbb{J}$ is Kähler

We suppose throughout this section that (M, g, J) is a strictly almost-Kähler, Einstein, weakly $*$-Einstein 4-manifold. Refreshing our memory, this means that:

$$P = p = 0, \quad q = -\overline{w^+_{00}}.$$

Using

$$\overline{\nabla}(\beta v \otimes \tilde{u} \otimes \tilde{u}^2) = \overline{\nabla}(\beta v) \otimes \tilde{u}^3 - 3\phi \otimes \beta v \otimes \tilde{u}^3,$$

we may write:

$$\begin{aligned}
\overline{\nabla}\beta v &= (q_1 v^2 + q_2 \frac{(v \otimes \tilde{v} + \tilde{v} \otimes v)}{2} + q_3 \tilde{v}^2) \otimes \tilde{u} \\
&\quad - \overline{w^+_{00}} \frac{(v \otimes \tilde{v} - \tilde{v} \otimes v)}{2} \otimes \tilde{u} + 3\phi \otimes \beta v \\
&= q_1 v \otimes \tilde{u} \otimes v \\
&\quad + \left(\frac{q_2 - \overline{w^+_{00}}}{2} \right) v \otimes \tilde{u} \otimes \tilde{v} + \left(\frac{q_2 + \overline{w^+_{00}}}{2} \right) \tilde{v} \otimes \tilde{u} \otimes v \\
&\quad + q_3 \tilde{v} \otimes \tilde{u} \otimes \tilde{v} + 3\phi \otimes \beta v.
\end{aligned}$$

We can, therefore, write:

$$\begin{aligned}\overline{\nabla}|\beta|^2 &= \overline{\nabla}(\beta v, \beta \tilde{v}) \\ &= q_1 \beta v \otimes \tilde{u} - q_1 \beta \tilde{v} \otimes u \\ &\quad + \left(\frac{q_2 + \overline{w^+_{00}}}{2}\right) \beta \tilde{v} \otimes \tilde{u} + \left(\frac{q_2 + \overline{w^+_{00}}}{2}\right) \beta v \otimes u.\end{aligned}$$

But by Lemma 3.3.1, we have that $\beta^2 = \frac{s^* - s}{16}$, and we have calculated $\mathrm{d}s^*$ in equation (3.12). Hence:

$$\mathrm{d}|\beta|^2 = w^+_{00} \beta v \otimes u + \overline{w^+_{00}} \beta \tilde{v} \otimes \tilde{u}.$$

Comparing these two formulae we see that $q_1 = 0$ and $q_2 = \overline{w^+_{00}}$ and hence:

$$\overline{\nabla} \beta v = \overline{w^+_{00}} \tilde{v} \otimes \tilde{u} \otimes v + q_3 \tilde{v} \otimes \tilde{u} \otimes \tilde{v} + 3\phi \otimes \beta v. \tag{3.16}$$

We should caution that one of the "miraculous" cancellations has taken. Instead of there being no $v \otimes \tilde{u} \otimes \tilde{v}$ term one might reasonably expect that we would get such a term, with coefficient proportional to $\overline{w^+_{00}}$. Were this cancellation not to occur, then one could run through the rest of the proof in this section and one would find that $\mathbb{J}$ is *never* Kähler. But since in Tod's examples, $\mathbb{J}$ is always Kähler we would then have a contradiction.

We wish to examine the equation $\mathrm{d}^2 s^* = 0$. To do this let us write,

$$A = a^1 v + a^2 \tilde{v}. \tag{3.17}$$

We compute $\nabla \mathrm{d}s^*$, choosing a gauge such that $\phi = 0$ at the given point. Using (3.12), (3.13) and (3.16) we get:

$$\begin{aligned}\frac{1}{16} \nabla \mathrm{d}s^* &= \beta A \otimes u \otimes v \otimes u + \frac{\beta^2 (3s^* - s)}{16} \tilde{v} \otimes \tilde{u} \otimes v \otimes u \\ &\quad + w^+_{00} \overline{w^+_{00}} \tilde{v} \otimes \tilde{u} \otimes v \otimes u + w^+_{00} q^3 \tilde{v} \otimes \tilde{u} \otimes \tilde{v} \otimes u \\ &\quad + w^+_{00} \beta^2 v \otimes \tilde{u} \otimes v \otimes \tilde{u} \\ &\quad + \text{conjugate}.\end{aligned}$$

Anti-symmetrising this we find:

$$\frac{1}{16} \mathrm{d}^2 s^* = -\beta a^2 u^2 - w^+_{00} q^3 \tilde{v}^2 + \text{conjugate}$$

So in a gauge with $\phi = 0$ we have $a^2 = 0$ and, if $w^+_{00} \neq 0$, $q^3 = 0$. We have already proved that we cannot have $w^+_{00} \equiv 0$ on our manifold, and so we must have $q^3 \equiv 0$. The equation $q^3 = 0$ is gauge invariant, but the equation $a^2 = 0$ is not. An equivalent gauge invariant expression is, by (3.17):

$$A = \alpha^1 v - 4w^+_{00}\Phi.$$

Where α^1 is a spinor of weight 3.

We deduce from (3.16) that

$$\overline{\nabla}\beta v = \overline{w^+_{00}}\tilde{v} \otimes \tilde{u} \otimes v - 3\Phi \otimes u \otimes \beta v + 3\tilde{\Phi} \otimes \tilde{u} \otimes \beta v \tag{3.18}$$

and from (3.7) and (3.13) we have:

$$\overline{\nabla} w^+_{00} = \alpha^1 v \otimes u - 4w^+_{00}\Phi \otimes u + \frac{(3s^* - s)}{16}\beta\tilde{v} \otimes \tilde{u} + 4w^+_{00}\tilde{\Phi} \otimes \tilde{u}. \tag{3.19}$$

Now $\beta^2 = \frac{s^* - s}{16}$ and

$$\frac{1}{16}\mathrm{d}s^* = w^+_{00}\beta v \otimes u + \overline{w^+_{00}}\beta\tilde{v} \otimes \tilde{u}.$$

Thus we have:

$$\nabla\beta = \frac{w^+_{00}}{2} v \otimes u - \frac{\overline{w^+_{00}}}{2}\tilde{v} \otimes \tilde{u}. \tag{3.20}$$

Combining this with (3.18) we obtain:

$$\nabla v = -\frac{w^+_{00}}{2\beta} v \otimes u \otimes v + \frac{\overline{w^+_{00}}}{2\beta}\tilde{v} \otimes \tilde{u} \otimes v + 3\Phi \otimes u \otimes v - 3\tilde{\Phi} \otimes \tilde{u} \otimes v. \tag{3.21}$$

In other words,

$$\nabla v = \psi \otimes v \tag{3.22}$$

where

$$\psi = -\frac{w^+_{00}}{2\beta} v \otimes u + \frac{\overline{w^+_{00}}}{2\beta}\tilde{v} \otimes \tilde{u} + 3\phi. \tag{3.23}$$

Thus the almost-complex structure $\mathbb{J}$ associated to v is Kähler.

Note that if the "miraculous" cancellation alluded to earlier had not occurred, we would get a term proportional to $\frac{w^+_{00}}{\beta} v \otimes \tilde{u} \otimes \tilde{v}$ in equation 3.21. This would mean that $\mathbb{J}$ could be Kähler if and only if $w^+_{00} \equiv 0$ contradicting Tod's examples.

3.4.2 $s = 0$

Proving that $s = 0$ is rather easy now. We calculate $\nabla\psi$ using (3.23).

$$\begin{aligned}\nabla\psi \quad = \quad & -\frac{\alpha^1}{2\beta} v\otimes u\otimes v\otimes u - \frac{3s^*-s}{32}\tilde{v}\otimes\tilde{u}\otimes v\otimes u \\ & + \frac{w^+_{00}}{2\beta^2}\left(-\frac{w^+_{00}}{2}v\otimes u + \frac{\overline{w^+_{00}}}{2}\tilde{v}\otimes\tilde{u}\right)\otimes v\otimes u \\ & - \frac{w^+_{00}}{2\beta}\psi\otimes v\otimes u - \frac{w^+_{00}}{2\beta}(\beta v\otimes\tilde{u}\otimes v\otimes\tilde{u}) \\ & - \text{conjugate} \\ & + 3\mathrm{d}\phi.\end{aligned}$$

Simplifying this and projecting onto $\bigwedge^2$ we find

$$(\nabla\psi)^{\bigwedge^2} = \left(\frac{3s^*-s}{16} + 3\left(\beta^2 - \frac{s^*}{8}\right)\right)u\tilde{u} + \left(\frac{3s^*-s}{16} - 3\beta^2\right)v\tilde{v}.$$

On the other hand since $\mathbb{J}$ is Kähler, we know by Lemma 3.3.1 that

$$(\nabla\psi)^{\bigwedge^2} = -\frac{s}{8}v\tilde{v}.$$

Since $\beta^2 = \frac{s^*-s}{16}$, we can equate the $u\tilde{u}$ terms in our expressions for $(\nabla\psi)^{\bigwedge^2}$ to find:

$$\frac{3s^*-s}{16} + 3\left(\frac{s^*-s}{16} - \frac{s^*}{8}\right) = 0.$$

This simplifies to give $s = 0$.

A corollary of this result is that a strictly almost-Kähler, Einstein, weakly $*$-Einstein manifold can never be compact — use Theorem 2.2.9.

Thus $\mathbb{J}$ defines a hyperkähler structure and hence $\mathrm{d}\psi = 0$. Thus we can now make a choice of gauge such that $\psi \equiv 0$ (This choice is unique up to multiplication by a constant in S^1). We rewrite our formulae in this gauge. Equation (3.23) becomes:

$$\phi = \frac{w^+_{00}}{6\beta}v\otimes u - \frac{w^+_{00}}{6\beta}\tilde{v}\otimes\tilde{u}. \tag{3.24}$$

or equivalently $\Phi = \frac{w^+_{00}}{6\beta}v$. Summing up our progress so far we have from (3.24) together with (2.1) that:

$$\nabla u = \frac{w^+_{00}}{6\beta}v\otimes u\otimes u - \frac{\overline{w^+_{00}}}{6\beta}\tilde{v}\otimes\tilde{u}\otimes u + \beta v\otimes\tilde{u}\otimes\tilde{u}, \tag{3.25}$$

from (3.22) we see that

$$\nabla v = 0, \tag{3.26}$$

from (3.20) we have

$$\nabla \beta = \frac{w^+_{00}}{2} v \otimes u + \frac{\overline{w^+_{00}}}{2} \tilde{v} \otimes \tilde{u}, \tag{3.27}$$

and from (3.19) we have

$$\nabla w^+_{00} = \left(\alpha^2 - \frac{2}{3} \frac{(w^+_{00})^2}{\beta} \right) v \otimes u + \left(3\beta^3 + \frac{2}{3} \frac{|w^+_{00}|^2}{\beta} \right) \tilde{v} \otimes \tilde{u}. \tag{3.28}$$

3.4.3 $\mathcal{D}$ is spanned by commuting Killing vector fields

The distribution $\mathcal{D}$ is spanned by vectors of the form $\lambda u \otimes \tilde{v} - \overline{\lambda} \tilde{u} \otimes v$ where $\lambda \in \mathbb{C}$. Suppose that λ is associated in this way to a vector field X lying in the distribution. Suppose also that $\mathrm{d}\lambda$ at the given point is given by:

$$\mathrm{d}\lambda = \lambda^1 u \otimes v + \lambda^2 u \otimes \tilde{v} + \lambda^3 \tilde{u} \otimes v + \lambda^4 \tilde{u} \otimes \tilde{v}.$$

Then we wish to find the conditions on the λ^i which ensure that X is a Killing vector field. As is well known, [Bes87], a vector field is Killing if and only if its covariant derivative is antisymmetric. Thus we wish to find the components of ∇X in $S^2 T^* \cong S^2 V^+ \otimes S^2 V^- \oplus \mathbb{C}$. First we compute

$$\begin{aligned} \nabla(\lambda u \otimes \tilde{v}) \quad = \quad & (\lambda^1 u \otimes v + \lambda^2 u \otimes \tilde{v} + \lambda^3 \tilde{u} \otimes v + \lambda^4 \tilde{u} \otimes \tilde{v}) \otimes u \otimes \tilde{v} \\ & + \lambda \left(\frac{w^+_{00}}{6\beta} v \otimes u \otimes u \otimes \tilde{v} \right. \\ & \quad \left. - \frac{\overline{w^+_{00}}}{6\beta} \tilde{v} \otimes \tilde{u} \otimes u \otimes \tilde{v} + \beta v \otimes \tilde{u} \otimes \tilde{u} \otimes \tilde{v} \right). \end{aligned}$$

We find the components of this in $S^2 V^+ \otimes S^2 V^-$,

$$\begin{aligned} & (\nabla(\lambda u \otimes \tilde{v}))^{S^2 V^+ \otimes S^2 V^-} = \\ & \quad \lambda^1 u^2 \otimes v\tilde{v} + \lambda^2 u^2 \otimes \tilde{v}^2 + \lambda^3 u\tilde{u} \otimes v\tilde{v} + \lambda^4 u\tilde{u} \otimes \tilde{v}^2 \\ & \qquad + \frac{\lambda w^+_{00}}{6\beta} u^2 \otimes v\tilde{v} - \lambda \frac{\overline{w^+_{00}}}{6\beta} u\tilde{u} \otimes \tilde{v}^2 + \lambda\beta \tilde{u}^2 \otimes v\tilde{v}. \end{aligned}$$

We deduce that:

$$\begin{aligned}(\nabla X)^{S^2V^+\otimes S^2V^-} &= \lambda^1 u^2\otimes v\tilde{v} + \lambda^2 u^2\otimes \tilde{v}^2 + \lambda^3 u\tilde{u}\otimes v\tilde{v} + \lambda^4 u\tilde{u}\otimes \tilde{v}^2 \\ &\quad - \overline{\lambda^1}\tilde{u}^2\otimes v\tilde{v} + \overline{\lambda^2}\tilde{u}^2\otimes v^2 + \overline{\lambda^3}u\tilde{u}\otimes v\tilde{v} - \overline{\lambda^4}u\tilde{u}\otimes v^2 \\ &\quad + \lambda\frac{w^+_{00}}{6\beta}u^2\otimes v\tilde{v} - \lambda\frac{\overline{w^+_{00}}}{6\beta}u\tilde{u}\otimes\tilde{v}^2 + \lambda\beta\tilde{u}^2\otimes v\tilde{v} \\ &\quad - \overline{\lambda}\frac{\overline{w^+_{00}}}{6\beta}\tilde{u}^2\otimes v\tilde{v} + \overline{\lambda}\frac{w^+_{00}}{6\beta}u\tilde{u}\otimes v^2 - \overline{\lambda}\beta u^2\otimes v\tilde{v}.\end{aligned}$$

We deduce from this that X is Killing only if:

$$\left.\begin{aligned}\lambda^1 + \tfrac{\lambda w^+_{00}}{6\beta} - \overline{\lambda}\beta &= 0\\ \lambda^2 &= 0\\ \lambda^3 + \overline{\lambda}^3 &= 0\\ \lambda^4 - \lambda\tfrac{\overline{w^+_{00}}}{6\beta} &= 0\end{aligned}\right\} \qquad (3.29)$$

We also need to check that the component of ∇X in $\mathbb{R}$ is zero. This is easily done, and it turns out that we get nothing extra. Thus X is Killing if and only if the system of equations (3.29) hold.

As one would expect, insisting that X is Killing does not quite suffice to determine its derivative given its value at a point — after all it is conceivable that $\mathcal{D}$ could be spanned by non-commuting Killing vector fields in which case one would have three linearly independent (over $\mathbb{R}$, not $C^\infty(\mathbb{R})$) vector fields X, Y, Z which would have to be linearly dependent at each point. Thus we need to consider the complete problem of finding a pair of commuting Killing vector fields.

Suppose that Y is another Killing vector field which lies in $\mathcal{D}$ and which commutes with X. Let μ be the complex valued function that determines Y and suppose that

$$\mathrm{d}\mu = \mu^1 u\otimes v + \mu^2 u\otimes\tilde{v} + \mu^3\tilde{u}\otimes v + \mu^4\tilde{u}\otimes\tilde{v}.$$

The condition that Y is Killing is of course expressed by the system (3.29) with the λ's replaced by μ's. It is an easy matter to check that under the assumption that these equations hold, and using the fact $[X,Y] = \nabla_X Y - \nabla_Y X$, the only additional conditions arising from the fact that X and Y commute are $\lambda^3 = 0$ and $\mu^3 = 0$.

As explained earlier, if $\mathcal{D}$ is spanned by two commuting Killing vector fields then if we choose any X, Y at a point in $\mathcal{D}$, then it must be possible to extend these X and Y to commuting Killing vector fields. We conclude that $\mathcal{D}$ is spanned by commuting Killing vector fields if and only if the system of equations:

$$\left.\begin{array}{rcl} \lambda^1 + \lambda\frac{w^+_{00}}{6\beta} - \overline{\lambda}\beta & = & 0 \\ \lambda^2 & = & 0 \\ \lambda^3 & = & 0 \\ \lambda^4 - \lambda\frac{w^+_{00}}{6\beta} & = & 0 \end{array}\right\} \tag{3.30}$$

has solutions taking any given value of λ at each point. But this system of equations is of Frobenius type, and so the system has the desired solutions if and only if it is integrable. So all that remains is to check the integrability of the above system. A practical version of Frobenius' theorem in this context is that such a system is integrable if and only if the value of $\mathrm{dd}\lambda$ one computes formally from the system (3.30) is always zero.

(3.30) can be rewritten as:

$$\mathrm{d}\lambda = \left(-\frac{\lambda w^+_{00}}{6\beta} + \overline{\lambda}\beta\right)(u \otimes v) + \left(\frac{\overline{\lambda w^+_{00}}}{6\beta}\right)(\tilde{u} \otimes \tilde{v}). \tag{3.31}$$

We deduce using (3.25) and (3.26) that:

$$\begin{aligned} \nabla\mathrm{d}\lambda \;=\; & -\frac{w^+_{00}}{6\beta}\mathrm{d}\lambda \otimes u \otimes v + \beta\mathrm{d}\overline{\lambda} \otimes u \otimes v + \frac{w^+_{00}}{6\beta}\mathrm{d}\lambda \otimes \tilde{u} \otimes \tilde{v} \\ & - \frac{\lambda}{6\beta}\mathrm{d}w^+_{00} \otimes u \otimes v + \frac{\lambda}{6\beta}\mathrm{d}\overline{w^+_{00}} \otimes \tilde{u} \otimes \tilde{v} \\ & + \frac{\lambda w^+_{00}}{6\beta^2}\mathrm{d}\beta \otimes u \otimes v + \overline{\lambda}\mathrm{d}\beta \otimes u \otimes v - \frac{\overline{\lambda w^+_{00}}}{6\beta^2}\mathrm{d}\beta \otimes \tilde{u} \otimes \tilde{v} \\ & + \left(-\frac{\lambda w^+_{00}}{6\beta} + \overline{\lambda}\beta\right)\left(\frac{w^+_{00}}{6\beta} v \otimes u \otimes u \otimes v \right. \\ & \qquad \left. - \frac{w^+_{00}}{6\beta}\tilde{v} \otimes \tilde{u} \otimes u \otimes v + \beta v \otimes \tilde{u} \otimes \tilde{u} \otimes v\right) \\ & + \frac{\overline{\lambda w^+_{00}}}{6\beta}\left(\frac{\overline{w^+_{00}}}{6\beta}\tilde{v} \otimes \tilde{u} \otimes \tilde{u} \otimes \tilde{v} \right. \\ & \qquad \left. - \frac{w^+_{00}}{6\beta} v \otimes u \otimes \tilde{u} \otimes \tilde{v} + \beta\tilde{v} \otimes u \otimes u\tilde{v}\right). \end{aligned}$$

We can now plug in the values for $\mathrm{d}\beta$, $\mathrm{d}w^+_{00}$ and $\mathrm{d}\lambda$ from (3.27), (3.28) and

(3.31). At the same time we project onto $\bigwedge^2$ to find $\mathrm{d}\mathrm{d}\lambda$:

$$\begin{aligned}
\mathrm{d}\mathrm{d}\lambda &= \Bigg[\frac{w_{00}^+}{6\beta}\left(\frac{\overline{\lambda w_{00}^+}}{6\beta}\right) - \beta\left(-\frac{\overline{\lambda w_{00}^+}}{6\beta} + \lambda\beta\right) + \frac{\overline{w_{00}^+}}{6\beta}\left(-\frac{\lambda w_{00}^+}{6\beta} + \overline{\lambda}\beta\right) \\
&\quad + \frac{\lambda}{6\beta}\left(3\beta^3 + \frac{2}{3}\frac{|w_{00}^+|^2}{\beta}\right) + \frac{\lambda}{6\beta}\left(3\beta^3 + \frac{2}{3}\frac{|w_{00}^+|^2}{\beta}\right) \\
&\quad - \frac{\lambda w_{00}^+}{6\beta^2}\left(\frac{\overline{w_{00}^+}}{2}\right) - \frac{\overline{\overline{\lambda} w_{00}^+}}{2} - \frac{\overline{\lambda w_{00}^+}}{6\beta^2}\left(\frac{w_{00}^+}{2}\right) \\
&\quad + \left(-\frac{\lambda w_{00}^+}{6\beta} + \overline{\lambda}\beta\right)\left(\frac{\overline{w_{00}^+}}{6\beta}\right) + \left(\frac{\overline{\lambda w_{00}^+}}{6\beta}\right)\left(-\frac{w_{00}^+}{6\beta}\right)\Bigg](u\tilde{u} + v\tilde{v}).
\end{aligned}$$

We simplify this:

$$\begin{aligned}
\mathrm{d}\mathrm{d}\lambda &= \Bigg[\frac{\lambda|w_{00}^+|^2}{36\beta^2} + \frac{\overline{\lambda w_{00}^+}}{6} - \lambda\beta^2 - \frac{\lambda|w_{00}^+|^2}{36\beta^2} + \frac{\overline{\overline{\lambda} w_{00}^+}}{6} \\
&\quad + \frac{\lambda\beta^2}{2} + \frac{\lambda|w_{00}^+|^2}{9\beta^2} + \frac{\lambda\beta^2}{2} + \frac{\lambda|w_{00}^+|^2}{9\beta^2} \\
&\quad - \frac{\lambda|w_{00}^+|^2}{12\beta^2} - \frac{\overline{\lambda w_{00}^+}}{2} - \frac{\lambda|w_{00}^+|^2}{12\beta^2} \\
&\quad - \frac{\lambda|w_{00}^+|^2}{36\beta^2} + \frac{\overline{\lambda w_{00}^+}}{6} - \frac{\lambda|w_{00}^+|^2}{36\beta^2}\Bigg](u\tilde{u} + v\tilde{v}) \\
&= \Bigg[\left(\frac{1}{36} - \frac{1}{36} + \frac{1}{9} + \frac{1}{9} - \frac{1}{12} - \frac{1}{12} - \frac{1}{36} - \frac{1}{36}\right)\frac{\lambda|w_{00}^+|^2}{\beta^2} \\
&\quad + \left(\frac{1}{6} + \frac{1}{6} - \frac{1}{2} + \frac{1}{6}\right)\frac{\overline{\overline{\lambda} w_{00}^+}}{6} \\
&\quad \left(-1 + \frac{1}{2} + \frac{1}{2}\right)\lambda\beta^2\Bigg](u\tilde{u} + v\tilde{v}) \\
&= 0.
\end{aligned}$$

Thus we conclude that on a strictly almost-Kähler, Einstein, weakly $*$-Einstein 4-manifold, $\mathcal{D}$ is always spanned by commuting Killing vector fields. As pointed out before, the rather implausible looking cancellation that has just occurred can be explained by observing that if the cancellation hadn't occurred we would instead have got an integrability condition depending on w_{00}^+ and β. Since these can take any values in Tod's examples, the integrability condition must always be satisfied. It is at this point that we use the fact that the vector fields commute. Our argument relies on the fact that M admits commuting Killing vector fields in $\mathcal{D}$ *only if* the system 3.30 is integrable. Of course, the explicit calculation provides a satisfying check on all our arguments so far.

It is easy to check, and rather obvious, that the Killing vectors we have just constructed necessarily preserve J and $\mathbb{J}$. Let us pick one of them, and call it X. Then X preserves $\mathbb{J}$, and the manifold is hyperkähler with the reverse orientation. The almost-complex structure J is determined from $\mathbb{J}$ via X and is almost-Kähler. Thus if X is a translational Killing vector field, we are precisely in the situation of Tod's construction. As we saw earlier, one cannot generalise Tod's examples to the case of rotational Killing vector fields, thus X must indeed be translational. This completes the proof.

3.5 Higher dimensions

Similar strategies to that used in four dimensions can be devised in higher dimensions to answer the question of whether or not a given Riemannian manifold locally admits an almost-Kähler structure. The nature of the strategy is the same: gradually build up a 0-jet, then a 1-jet, etc. solution of the problem until an algebraic obstruction to existence is found or until we are in a position to apply Frobenius' theorem. To illustrate the idea we shall prove that hyperbolic space of dimensions 6 and above does not, even locally, admit a compatible almost-Kähler structure. In dimension 6, this is a new result, but the result was proved in dimensions 8 and above in [Ols78].

The starting point is once again the Ricci identity:

$$\alpha(\nabla\nabla\omega) = R\omega.$$

We remind ourselves of the analysis of this given in [FFS94] — though we shall only be concerned with the almost-Kähler case. They rewrite the Ricci identity as:

$$\alpha(\overline{\nabla}\xi)(X,Y) + \alpha(\beta(\xi \odot \xi))(X,Y) = R_{X,Y}\omega$$

where

$$\alpha : T^* \otimes T^* \otimes \bigwedge^2 \longrightarrow \bigwedge^2 \otimes \bigwedge^2$$

by antisymmetrisation and

$$\beta : T^* \otimes [\![\bigwedge^{2,0}]\!] \otimes T^* \otimes [\![\bigwedge^{2,0}]\!] \longrightarrow T^* \otimes T^*[\![\bigwedge^{2,0}]\!]$$

by contracting the first $[\![\bigwedge^{2,0}]\!] \cong \mathfrak{u}(n)^\perp$ component with the second T^*M. Clearly $\alpha(\overline{\nabla}\xi)(X,Y)$, $\alpha(\beta(\xi \odot \xi))$ and $R\omega$ lie in $\bigwedge^2 \otimes [\![\bigwedge^{2,0}]\!] \cong \mathcal{K}^\perp$. Thus we are able to equate the components of R in $\mathcal{K}^\perp$ with components of $\xi \odot \xi$ and $\overline{\nabla}\xi$.

Now according to [FFS94], in dimensions 8 and above $\mathcal{K}^\perp$ splits into 7 components which they named as follows.

$$\mathcal{K}^\perp = \mathcal{K}_{-1} \oplus \mathcal{K}_{-2} \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8.$$

Most important to us are $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$ and $\mathcal{C}_4$ which are isomorphic to $\mathbb{R}$, $\bigwedge_0^{1,1}$, and $\bigwedge_0^{2,2}$ respectively.

We wish to know which components are determined by $\xi \odot \xi$, which by $\overline{\nabla}\xi$ and which by both. The answer is:

Lemma 3.5.1 *If (M^{2n}, g, J) is an almost-Kähler manifold with $m \geq 4$ then each of the tensors $\alpha\overline{\nabla}\xi$ and $\alpha \circ \beta(\xi \odot \xi)$ contributes to the components of R in $\mathcal{K}^\perp$ iff there is a tick in the corresponding box in the table below.*

	$\mathcal{K}_{-1}$	$\mathcal{K}_{-2}$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$
$\alpha(\overline{\nabla}\xi)$				$\surd$	$\surd$	$\surd$	$\surd$
$\alpha \circ \beta(\xi \odot \xi)$	$\surd$	$\surd$	$\surd$				

Proof: *All one has to do is recall that on an almost-Kähler manifold $\xi \in \mathcal{W}_2$, decompose everything into irreducibles and apply Schur's lemma to find that there couldn't possibly be ticks except where we have put them. One then only has to check that there are ticks where we have put them — this is an easy calculation.*

A nice shortcut to the Schur's lemma calculation is to use the fact that $\mathrm{U}(n) \cong \mathrm{S}^1 \times \mathrm{SU}(n)$ and so we can associate a weights in $\mathbb{Z}$ to any representation of $\mathrm{U}(n)$ by taking the weights with which the S^1 component acts. Thus $\bigwedge^{1,0}$ has weight 1, $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$ has weight 3 and so $\mathcal{W}_2 \subseteq [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ has weights 3 and -3. So $\xi \odot \xi$ can only have components of weights 6, 0 and -6 whereas $\overline{\nabla}\xi$ can only have components of weights 4, 2, -2 and -4. The weights of the components of R are similarly easy to calculate and one sees that we shall get the table above. $\square$

Much the same results follow in the 6 dimensional case. The only difference is that R does not have a $\mathcal{C}_4$ component in this case. The table still holds so long as one ignores that column.

3.5.1 6-dimensional hyperbolic space

Let (M^6, g, J) be an almost-Kähler, constant curvature, 6 manifold. By the table in [FFS94] we see that the component of $\xi \otimes \xi$ in $\mathcal{K}_{-2}$ must vanish, and also $\|\xi\|^2$ must be some constant multiple of the scalar curvature. As a first step, we must try to identify the space of such ξ.

We shall call the relevant space (at a point x) P_x. So

$$P_x := \{\xi \in \mathcal{W}_2 : \alpha\beta(\xi \otimes \xi) = c\mathbf{1}\}$$

where $\mathbf{1}$ denotes the identity map $\mathbf{1} : \mathfrak{u}(3)^\perp \longrightarrow \mathfrak{u}(3)^\perp$, and where c is some constant multiple of the scalar curvature. Since $\|\xi\|^2$ is a constant (in fact negative) multiple of the scalar curvature, we may assume without loss of generality that $c = -1$.

To state the next lemma, we need to pick some $\eta \in \bigwedge^{3,0}$ with constant norm. This gives us a reduction of the structure group to SU(3). This allows us to define an SU(3) equivariant map ϕ by the composition:

$$\bigwedge^{1,0} \otimes \bigwedge^{2,0} \xrightarrow{\wedge\omega} \bigwedge^{1,0} \otimes \bigwedge^{3,1} \xrightarrow{\eta^*} \bigwedge^{1,0} \otimes \bigwedge^{0,1} \cong \bigwedge^{1,1} \xrightarrow{\mathbb{R}} T^*_{\mathbb{R}}M \otimes T^*_{\mathbb{R}}(M)$$

where η^* is the adjoint of wedging with η and $\mathbb{R}$ is the inclusion of the complex vector space in the real one.

Lemma 3.5.2 *$\xi \in P_x$ implies that $\phi(\xi) \in \mathrm{U}(3) \subseteq \mathrm{SO}(6) \subseteq T^*M \otimes TM \cong T^*M \otimes T^*M$.*

Proof: *Write $* : \bigwedge^{0,1} \longrightarrow \bigwedge^{2,0}$ for the inverse of $\eta^* \circ (\wedge\omega)$. If X_1, X_2, X_3 gives a unitary basis for $\bigwedge^{1,0}$, then since $\xi \in \mathcal{W}_2 \subseteq [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ we can write $E = a^{ij} X_i \otimes *\overline{X}_j$ for the component of ξ in $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$. The condition that $\xi \in \mathcal{W}_2$ can now be written $\sum a^{ii} = 0$. We have*

$$\begin{aligned} E = \; & a^{11} X_1 \otimes X_2 \wedge X_3 + a^{21} X_2 \otimes X_2 \wedge X_3 + a^{31} X_3 \otimes X_2 \wedge X_3 \\ & + a^{12} X_1 \otimes X_3 \wedge X_1 + a^{22} X_2 \otimes X_3 \wedge X_1 + a^{32} X_3 \otimes X_3 \wedge X_1 \\ & + a^{13} X_1 \otimes X_1 \wedge X_2 + a^{23} X_2 \otimes X_1 \wedge X_2 + a^{33} X_3 \otimes X_1 \wedge X_2. \end{aligned}$$

So if we write $\alpha : \bigwedge^{1,0} \otimes \bigwedge^{2,0} \longrightarrow \bigwedge^2 \otimes \bigwedge^{1,0}$ by anti-symmetrisation. Then

$$\begin{aligned} \alpha E = \; & a^{11} * \overline{X}_3 \otimes X_3 - a^{21} * \overline{X}_1 \otimes X_2 - a^{31} * \overline{X}_1 \otimes X_3 \\ & + a^{11} * \overline{X}_2 \otimes X_2 + a^{22} * \overline{X}_1 \otimes X_1 - a^{32} * \overline{X}_2 \otimes X_3 \\ & - a^{12} * \overline{X}_2 \otimes X_1 + a^{22} * \overline{X}_3 \otimes X_3 + a^{33} * \overline{X}_1 \otimes X_1 \\ & - a^{13} * \overline{X}_3 \otimes X_1 + a^{23} * \overline{X}_3 \otimes X_2 + a^{33} * \overline{X}_2 \otimes X_2. \end{aligned}$$

So using the fact that $a^{11}+a^{22}+a^{33}=0$ *we have:*

$$E := a^{ij} X_i \otimes *\overline{X}_j,$$

$$\alpha E := -a^{ji} * \overline{X}_i \otimes X_j.$$

Recall that the map β *is defined in [FFS94] by*

$$\beta : (T^*M \otimes \mathfrak{u}(3)^\perp) \otimes (T^*M \otimes \mathfrak{u}(3)^\perp) \longrightarrow T^*M \otimes T^*M \otimes \mathfrak{u}(3)^\perp$$

by contracting the first $\mathfrak{u}(3)^\perp$ *with the second* T^*M*. Hence* $\alpha(\beta(\xi\otimes\xi)) = \beta(\alpha(\xi)\otimes\xi)$*. So the condition that* $\alpha(\beta(\xi\otimes\xi)) = -\mathbf{1}$ *implies* $\beta(\alpha E \otimes \overline{E}) = -\mathbf{1}$ *and hence that* ξ *is in* $\mathrm{U}(3)$*.* $\square$

We shall also use

Lemma 3.5.3 *If* $A \in T^*M \otimes TM$ *and* A *is positive definite, and if we define*

$$V_A := \{W \in T^*M \otimes T^*M \otimes TM : W_X^T A + A^T W_X = 0 \quad \forall X\}$$

and if $\alpha : T^*M \otimes T^*M \otimes TM \longrightarrow \bigwedge^2 \otimes TM$ *by anti-symmetrisation, then* $V_A \cap \operatorname{Ker}\alpha = \{0\}$*.*

Proof: If we use A *to define an isomorphism between* TM *and* T^*M *then under this identification,*

$$V_A = \{W \in T^*M \otimes \bigwedge^2\}.$$

But the map $\bigwedge^2 \otimes \bigwedge^1 \hookrightarrow \bigwedge^1 \otimes \bigwedge^1 \otimes \bigwedge^1 \xrightarrow{\alpha} \bigwedge^1 \otimes \bigwedge^2$ *is well known to be an isomorphism. Hence the result follows.* $\square$

Theorem 3.5.4 *If* (M^6, g, J) *is a constant curvature, almost-Kähler* 6*-manifold, then it is necessarily flat and hence Kähler.*

Proof: Suppose that M^6 *is constant curvature and almost-Kähler. Then we must have that* $\xi \in P_x$ *at every point. Hence* $\overline{\nabla}_X \xi$ *must lie in* $T_\xi P_x$ *for all vectors* X*. We have already shown that* P_x *lies in* $\mathrm{SO}(6)$ *and so we must have that* $\phi(\overline{\nabla}_X \xi) \in T_{\phi(\xi)}\mathrm{SO}(6)$*. Let us write* $B = \phi(\overline{\nabla}\xi)$ *and* $A = \phi(\xi)$*. So* $B_X \in T_A\mathrm{SO}(6)$*. Equivalently* $B \in V_A$*. On the other hand, by the table in* *the previous section we must have that $\alpha(\overline{\nabla}\xi) = 0$. So by Lemma 3.5.3 we must have that $\overline{\nabla}\xi = 0$.*

If we now apply the Ricci identity to ξ, we see that $\overline{R}\xi = 0$, where $\overline{R}$ is the curvature of $\overline{\nabla}$. On the other hand the 6-dimensional analogue of Lemma 3.3.1 readily shows that:

$$\overline{R} = \left(\begin{array}{c|c|c} a & * & * \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & * \end{array} \right)$$

where the $*$'s denote potentially non-zero terms, a is a non-zero multiple of the scalar curvature and where we have block decomposed $\overline{R}$ with respect to the splitting $\bigwedge^2 = \langle \omega \rangle \oplus [\![\bigwedge^{2,0}]\!] \oplus [\bigwedge_0^{1,1}]$. So

$$\overline{R}\xi \in \langle \omega \rangle \otimes \mathcal{W}_2 \oplus [\![\bigwedge^{2,0}]\!] \otimes \mathcal{W}_2 \oplus [\bigwedge_0^{1,1}] \otimes \mathcal{W}_2.$$

Moreover, the component in $\langle \omega \rangle \otimes \mathcal{W}_2$ is equal to $a\xi$. So we get a contradiction unless the scalar curvature is identically zero. □

3.5.2 Hyperbolic space in dimensions 8 and above

Let (M^{2m}, g, J) be an almost-Kähler manifold, $m \geq 4$.

If $\xi \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ then we may write

$$\xi = \eta_{ijk} \mathrm{d}z^i \otimes \mathrm{d}z^j \wedge \mathrm{d}z^k + \overline{\eta}_{ijk} \mathrm{d}\overline{z}^i \otimes \mathrm{d}\overline{z}^j \wedge \mathrm{d}\overline{z}^k$$

where $\eta_{ijk} \in \mathbb{C}$ and $\eta_{ijk} = -\eta_{ikj}$ (We are using the convention that one should sum over repeated indices). Now $\xi \in [\![A]\!]$ iff it lies in the kernel of the anti-symmetrisation map

$$a : [\![\bigwedge^{0,1} \otimes \bigwedge^{0,2}]\!] \longrightarrow [\![\bigwedge^{0,3}]\!]$$

Equivalently iff $\eta_{ijk} + \eta_{jki} + \eta_{kij} = 0$.

Now,

$$\begin{aligned} \alpha\beta(\xi \odot \xi) &= \eta_{(ij)k} \overline{\eta}_{klm} \mathrm{d}z^i \wedge \mathrm{d}z^j \otimes \mathrm{d}\overline{z}^l \wedge \mathrm{d}\overline{z}^m \\ &\quad + \overline{\eta}_{(ij)k} \eta_{klm} \mathrm{d}\overline{z}^i \wedge \mathrm{d}\overline{z}^j \otimes \mathrm{d}z^l \wedge \mathrm{d}z^m \end{aligned}$$

By the symmetries of η we have $\eta_{(ij)k} = -2\eta_{kij}$, so

$$-\frac{1}{2}\alpha\beta(\xi \odot \xi) = \Re(\eta_{kij}\overline{\eta}_{klm}\mathrm{d}z^i \wedge \mathrm{d}z^j \otimes \mathrm{d}\overline{z}^l \wedge \mathrm{d}\overline{z}^m)$$

So $\alpha\beta(\xi \odot \xi) \in [\bigwedge^{2,0} \otimes \bigwedge^{0,2}] \cong [\bigwedge^{2,2}] \cong [\bigwedge_0^{2,2} \oplus \bigwedge_0^{1,1} \oplus \mathbb{R}]$. We shall write E for the component of $\alpha\beta(\xi \odot \xi)$ in $[\bigwedge_0^{2,2} \oplus \bigwedge_0^{1,1}]$. We have that

$$E = \Re(\eta_{kij}\overline{\eta}_{klm}\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^l \wedge \overline{z}^m - C\eta_{abc}\overline{\eta}_{abc}\mathrm{d}z^d \wedge \mathrm{d}z^e \wedge \mathrm{d}\overline{z}^d \wedge \overline{z}^e)$$

for some constant C. Clearly $C \neq 0$.

Lemma 3.5.5 *In complex dimensions, m, greater than or equal to 4, $E = 0$ iff $\xi = 0$.*

Proof: Suppose $E = 0$. Write $\vec{\eta}_{ij}$ for the vector $(\eta_{1ij}, \eta_{2ij}, \ldots, \eta_{mij})$. Suppose that $\{i, j\} \neq \{l, m\}$ then taking the $\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^l \wedge \mathrm{d}\overline{z}^m$ component of the equation $E = 0$ we see that

$$\eta_{kij}\overline{\eta}_{klm} = 0$$

Equivalently $\langle \vec{\eta}_{ij}, \vec{\eta}_{lm} \rangle = 0$. So we have that $\{\vec{\eta}_{ij} : 1 \leq i < j \leq m\}$ is orthogonal. So if $\frac{m(m-1)}{2} > m$ — that is if $m > 3$ — then we see that at least one of the $\vec{\eta}_{ij}$ is zero. So suppose $\vec{\eta}_{lm} = 0$. Then calculating the $\mathrm{d}z^l \wedge \mathrm{d}z^m \wedge \mathrm{d}\overline{z}^l \wedge \mathrm{d}\overline{z}^m$ component of the equation $E = 0$ we get that

$$\eta_{klm}\overline{\eta}_{klm} - C\eta_{abc}\eta_{abc} = 0.$$

In other words,

$$\|\vec{\eta}_{lm}\|^2 - C \sum_{1 \leq i < j \leq m} \|\vec{\eta}_{ij}\|^2 = 0$$

Since $\vec{\eta}_{lm} = 0$ and $C \neq 0$ we must have that all the $\vec{\eta}_{ij}$ are zero. So $\xi = 0$.
$\square$

In conclusion,

Theorem 3.5.6 *If (M^{2m}, g, J) is an almost-Kähler manifold with $m \geq 4$ and if the components of R in $\mathcal{K}_{-2}$ and $\mathcal{C}_4$ are both zero then we must have that the manifold is Kähler.*

As a corollary we have:

Corollary 3.5.7 *[Ols78] If (M^{2m}, g, J) is an almost-Kähler manifold with $m \geq 4$ and if M has constant curvature then M is flat.*

Of course our theorem is marginally stronger than Olszak's original result. However, our real motivation for reproving this result is to show how it fits in with our strategy.

Notice that in dimensions 8 and above, a single application of the Ricci identity was sufficient to prove our result. In dimension 6 we required two applications of the Ricci identity, in dimension 4 the corresponding result requires three applications of the Ricci identity.

Chapter 4

Cartan–Kähler theory

4.1 Introduction

In this chapter we shall apply the ideas of Cartan–Kähler theory to the study of almost-Kähler manifolds. The principal point the author wishes to get across is that we have been tacitly using Cartan–Kähler theory throughout the thesis. Thus Cartan–Kähler theory provides a unifying perspective for the thesis. Indeed, Cartan–Kähler theory provides a useful conceptual framework for a large proportion of local differential geometry.

Although Cartan–Kähler theory is usually viewed as a tool for proving existence results about differential equations, we shall emphasise a different aspect. Specifically, Cartan–Kähler theory provides a systematic method to find non-obvious conditions which we shall "obstructions" that solutions to a differential equation must satisfy.

We shall find one such obstruction to finding almost-Kähler, Einstein metrics which will allow us to prove:

Theorem 4.1.1 *If (M^4, g, J) is a compact almost-Kähler, Einstein 4-manifold with $W^+_{00} \equiv 0$ then M is Kähler.*

Thus in fact we use Cartan–Kähler theory primarily to prove a non-existence result.

Nevertheless, the author's interest in Cartan–Kähler theory was sparked off by the idea of using it to prove that almost-Kähler, Einstein 4-metrics locally exist. Indeed, at one point the author believed that he had succeeded in doing this — however, a gap that proved impossible to fill was pointed out to the author. The motivation to pursue this line of enquiry further is somewhat diminished by the examples of Przanowski, Nurowski and Tod which show that almost-Kähler Einstein 4-metrics do exist. However, the question "Are there almost-Kähler, Einstein manifolds which are not given by Tod's construction?" is now a rather natural question to ask, and further examples would raise similar questions. Cartan–Kähler theory provides, in principle, a route to answering all such questions in one fell swoop. With this motivation we push the Cartan–Kähler theory as far as we can, but as the algebraic difficulties mount up we eventually have to give in.

Thus we are not, in the case of almost-Kähler Einstein 4-manifolds, able to use Cartan–Kähler theory to prove any existence results. However, we do extract valuable information. In particular we do show that if it is true, for example, that all almost-Kähler Einstein 4-metrics are given by Tod's construction then one would have to examine a rather high number of derivatives in order to prove it.

In the first section of this chapter we describe Cartan–Kähler theory. Our aim is to convey the general framework which Cartan–Kähler theory provides for viewing many local calculations in differential geometry — thus we shall glide over a number of technicalities. For a fully detailed account of Cartan–Kähler theory, the reader should consult the references. We use the view point of jet bundles and commutative algebra described in [Gol67a], [Gol67b] and the final chapters of [BCG$^+$91] as we find this approach more intuitive than the traditional approach using exterior differential systems (see [Car45] and the earlier chapters of [BCG$^+$91]). Our emphasis is on the "obstructions" rather than on the existence results exemplified by the Cartan–Kähler Theorem. It should be notied that what we call "obstructions" are called "curvature" by Goldschmidt and "torsion" in [BCG$^+$91].

The second section of the chapter contains the applications of Cartan–Kähler theory to almost-Kähler, Einstein manifolds that we have already discussed.

The third section of the chapter attempts to discuss Hermitian, Einstein 4-manifolds from the perspective of Cartan–Kähler theory. We include this material partly because the example of the Riemannian Goldberg-Sachs theorem provides a good example of the usefulness of the concept of "obstructions" and thus justifies further the material in the second section. In the case of Hermitian Einstein manifolds, we are able to carry the analysis through to the end. In fact we succeed in finding a description of Hermitian Einstein manifolds in terms of a differential equation which is patently formally integrable (i.e. which is unobstructed). Indeed in the case of Hermitian, Ricci-flat manifolds this equation is the SU(∞)-Toda field equation which we met in section 1.3.2. The connection between Hermitian, Einstein 4-metrics and the Toda field equation was first shown by Przanowski and Bialecki in [PB87]. Our proof, however, is simpler in that it avoids use of Lie–Bäcklund transformations. Our result in the case of Hermitian Einstein manifolds with non-zero scalar curvature would appear to be new. Since Hermitian manifolds are of only tangential interest to the thesis as a whole, we only describe the essential points of our proof and omit the details of the calculations.

4.2 Cartan–Kähler theory

4.2.1 Jets and differential operators

The basic idea behind Cartan–Kähler theory is a simple one: one tries to build up analytic solutions to the given differential equation order by order. For this reason we shall work throughout this chapter in the analytic category. For our applications this is no restriction since all Einstein metrics are analytic, [DK81]. However, one should remark that it is possible to extend certain elements of Cartan–Kähler theory to the smooth case.

In actual fact working with analytic solutions to a given order (i.e. polynomials) is rather cumbersome and for this reason we wish to use the language of jet bundles. Jet bundles encapsulate, in an invariant manner, the key facts about polynomials that we shall need.

Let M be a manifold and let $\pi : \mathcal{E} \longrightarrow M$ be a fibre bundle with fibre $\mathcal{E}_x$ over a point $x \in M$. If p is a point in $\mathcal{E}_x$ then let us write E_p for the tangent space of the fibre at p.

A 1-*jet* of $\mathcal{E}$ at x is essentially a possible value for the first order part of the Taylor series expansion of a section of $\mathcal{E}$ about x. Thus a 1-jet of $\mathcal{E}$ at x consists of a pair (p, ϕ) with:

- $p \in \mathcal{E}_x$
- $\phi \in T^*_x M \otimes E_p$.

We think of ϕ as a map $TM \longrightarrow E_p$ which says what the derivative of the section is at x.

We call the space of 1-jets of $\mathcal{E}$ at x $J_1(\mathcal{E})(x)$. We shall usually drop the x. Clearly we have a projection:

$$j_1 : \Gamma(\mathcal{E}) \longrightarrow J_1(\mathcal{E}),$$

where $\Gamma(\mathcal{E})$ is the space of sections of $\mathcal{E}$, given by taking the first order part of a section.

In exactly the same way, we can define k-jets to be the set of possible k-th order Taylor expansions of sections. Clearly we have a projection $j_k : \Gamma(\mathcal{E}) \longrightarrow J_k(\mathcal{E})$. We can now define a differential operator.

Definition 4.2.1 *A differential operator $D : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{F})$ is a map of the form $(De)(x) = (\phi(j_k(e)))(x)$ where ϕ is a bundle map from $J_k(\mathcal{E})$ to $\mathcal{F}$.*

Given a section e of $\mathcal{E}$ we define the bundle $E_e = \coprod_{x \in X} E_e(x)$ whose sections represent infinitesimal perturbations of e. Thus if $De = f$, we can define the linearisation of D at e to be a k-th order operator from E_e to F_f.

The crucial point about jet-bundles is the following exact sequence:

$$0 \longrightarrow S^l T^* \otimes E \xrightarrow{\iota} J_l(E) \xrightarrow{\pi} J_{l-1}(E) \longrightarrow 0 \tag{4.1}$$

which states, in effect, that an l-jet is composed of an element of $S^l T^* \otimes E$ which represents its top order term and an $(l-1)$-jet. The fact that one only requires an element of $S^l T^* \otimes E$ rather that $(\bigotimes^l T^*) \otimes E$ is an expression of the fact that derivatives in different directions commute. It is this that leads to all the rich algebra which follows and explains why the study of series solutions of differential equations is so much more subtle in dimensions greater than one.

4.2.2 Prolongation

Given a map $\phi : J_k(\mathcal{E}) \longrightarrow \mathcal{F}$ one can define its *i-th prolongation* to be the map $p_i(\phi) : J_{k+i}(\mathcal{E}) \longrightarrow J_i(\mathcal{F})$ given by:

$$p_i(\phi)(j_{k+i}(e)) = j_i(\phi(j_k(e)))$$

for any section e of $\mathcal{E}$. The i-th prolongation $p_i(D) : \Gamma(\mathcal{E}) \longrightarrow \Gamma(J_i(\mathcal{F}))$ is defined in the same way. It should be though of as mapping a section e to De and the first i derivatives of De.

Given a differential equation of the form

$$De = f \tag{4.2}$$

where $f \in \Gamma(\mathcal{F})$, let us write $N_k \subseteq J_k(\mathcal{E})$ for the space of k-jets, u, satisfying $\phi(u) = f$ where ϕ is the bundle map associated to D. Similarly let us write N_{k+i} for the space of $(k+i)$-jets, u, satisfying $p_i(\phi(u)) = j_i(f)$.

Definition 4.2.2 *N_{k+i} is the space of all $(k+i)$-jet* solutions *to the differential equation (4.2).*

Let $\pi : N_{k+i+1} \longrightarrow N_{k+i}$ be the restriction of the projection

$$\pi : J_{k+i+1}(\mathcal{E}) \longrightarrow J_{k+i}(\mathcal{E}).$$

Definition 4.2.3 *The equation (4.2) is said to be* formally integrable *if $\pi : N_{k+i+1} \longrightarrow N_{k+i}$ is surjective for all $i \geq 0$.*

(In actual fact one should also insist that the dimension of N_{k+i+1} is locally constant — we shall assume throughout that the dimensions of all the bundles we consider are locally constant. This allows us to ignore some rather tedious technicalities.)

The interpretation of formal integrability is that a k-th order power series solution can be extended to a power series solution of any desired order. Thus, modulo questions of convergence, solutions to the differential equation must exist.

Definition 4.2.4 *If $N_{k+i+1} \longrightarrow N_{k+i}$ is not surjective, we shall say that there is an* obstruction *to extending $(k+i)$-jet solutions to $(k+i+1)$-jet solutions. We shall refer to the condition that a $(k+i)$-jet must satisfy in order to be extended as a* hidden *condition.*

The motivation for these definitions is that if $N_{k+i+1} \longrightarrow N_{k+i}$ is not surjective then one can find $(k+i)$-th order solutions to the differential equation which *cannot* be extended to bona-fide solutions. Thus one obtains $(k+i)$-th order equations that any solutions must satisfy which cannot be found except by considering $(k+i+1)$-jets.

4.2.3 The symbol and obstructions

In trying to understand power series solutions to differential equations, the most important part of a differential operator is its top order term. An invariant way of describing this top order term is given by the *symbol*.

Definition 4.2.5 *If $D : \Gamma(E) \longrightarrow \Gamma(F)$ is a linear k-th order operator then its symbol*

$$\sigma(D) : S^k T^* \otimes E \longrightarrow F$$

is defined by

$$\sigma(D) = \phi \circ \iota : S^k T^* \otimes E \longrightarrow F$$

where ι comes from the exact sequence (4.1).

The l-th prolongation of the symbol is the map

$$\sigma_l(\phi) := (\mathbf{1} \otimes \sigma) \circ \iota : S^{k+l} T^* \otimes E \longrightarrow S^l T^* \otimes F$$

where $\iota : S^{k+l} T^* \otimes E \longrightarrow S^l T^* \otimes S^k T^* \otimes E$ is the natural inclusion.

The importance of the symbol is clear. If the symbol is onto then we can always find k-jet solutions to our equation by picking any k-jet at all and modifying its top order term appropriately. Similar remarks apply to the prolongations of the symbol.

To see this more clearly, let us write R_{k+i} for the space of $(k+i)$-jet solutions to a linear equation $\nabla e = 0$. Let us write g_{k+i} for the kernel of $\sigma_i(D)$ and W for the cokernel.

Now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_{k+i} & \longrightarrow & S^{k+i}T^* \otimes E & @>\sigma_i(D)>> & S^iT^* \otimes F & \longrightarrow & W \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_{k+i} & \longrightarrow & J_{k+i}(E) & @>p_i(D)>> & J_i(F) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_{k+i-1} & \longrightarrow & J_{k+i-1}(E) & @>p_{i-1}(D)>> & J_{i-1}(F) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

All the rows and columns are exact except for the first column, and it is the homology of this column that we wish to know about. By standard diagram chasing we can define a "zig-zag" map Ω from R_{k+i-1} to W with the property that $u \in R_{k+i-1}$ satisfies $\Omega u = 0$ if and only if there exists $v \in R_{k+i}$ above u.

Thus if $\sigma_i(D)$ is onto, $R_{k+i} \longrightarrow R_{k+i-1}$ is onto. Otherwise the equation $\Omega u = 0$ is a hidden equation.

One need not restrict one's attention to linear differential equations to make this interpretation of the symbol. To state the result we need one technical definition: if $f_1 : Z \longrightarrow X$ and $f_2 : Z \longrightarrow Y$ are maps and $\mathcal{E}_1$ is a bundle over X and $\mathcal{E}_2$ is a bundle over Y then we shall write $\mathcal{E}_1 \otimes_Z \mathcal{E}_2$ for the bundle over Z given by pulling back $\mathcal{E}_1$ and $\mathcal{E}_2$ and then taking their tensor product. We shall write $V(\mathcal{E})$ for the vector bundle over $\mathcal{E}$ whose fibre at each point $e \in E$ is the space of tangent vectors in the direction of the fibre at e.

One can easily prove:

Proposition 4.2.6 *[Gol72] Suppose D is a differential operator of order k, $D : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{F})$ and suppose that $f \in \Gamma(\mathcal{F})$. If we have an exact sequence of vector bundles:*

$$S^{k+l}T^* \otimes_{N_{k+l-1}} V(\mathcal{E}) @>\sigma_l(\delta D)>> S^lT^* \otimes N_{k+l-1}V(\mathcal{F}) \xrightarrow{\tau} W \longrightarrow 0$$

then we can define $\Omega : N_{k+l-1} \longrightarrow W$ such that

$$N_{k+l} @>\pi>> N_{k+l-1} @>\Omega>> W$$

is exact.

The only additional ingredient one requires to prove this is some technical material on affine bundles and the interpretation of jet-bundles as affine bundles. One can find this in [Gol67b].

An example of an obstruction will help illuminate these ideas. Suppose that $f : E \longrightarrow T^* \otimes E$ is a linear map and that ∇ is a connection on a vector bundle E. Let us consider the differential equation:

$$\nabla e = f(e)$$

for a section e of E. We can always find a 1-jet solution $(e, \nabla e = f(e))$ to the equation. Now

$$\sigma(\nabla) : T^* \otimes E \longrightarrow T^* \otimes E$$

is of course the identity map and so $\sigma_1(\nabla) : S^2T^* \otimes E \longrightarrow T^* \otimes T^* \otimes E$ is just the inclusion.

Consider the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & S^2T^* \otimes E & @>\sigma_1(\nabla)>> & T^* \otimes T^* \otimes E & @>\pi>> & \bigwedge^2 \otimes E \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_2 & \longrightarrow & J_2(E) & @>p_1(\nabla - f(e))>> & J_1(T^* \otimes E) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_1 & \longrightarrow & J_1(E) & @>\nabla - f(e)>> & T^* \otimes E & &
\end{array}$$

If $(e, \nabla e)$ is a 1-jet solution. We wish to compute Ω of this 1-jet — i.e. we wish to compute the effect of the zig-zag map. First consider any 2-jet $(e, \nabla e, \nabla\nabla e) \in J_2(E)$ extending the 1-jet. Then

$$p_1(\nabla - f(e))(e, \nabla e, \nabla\nabla e) = (f(e), \nabla\nabla e - (\nabla f)e - f(\nabla e)) \in J_1(T^* \otimes E).$$

So the lift of this to $T^* \otimes T^* \otimes E$ is $\nabla\nabla e - (\nabla f)e - f(\nabla e)$. Projecting this onto $\bigwedge^2 \otimes E$ we obtain $\Omega(e, \nabla e) = \alpha(\nabla\nabla e - (\nabla f)e - f(\nabla e)) = Re - \alpha((\nabla f)e - f(\nabla e))$. Thus the 1-jet solution can be lifted to a 2-jet solution if and only if:

$$Re = \alpha((\nabla f)e - f(\nabla e)). \tag{4.3}$$

In words, the 1-jet can be lifted to a 2-jet solution if and only if "$\alpha(\nabla\nabla e)$ as calculated formally from the equation and the 1-jet is equal to the curvature applied to the 0-jet".

Of course, one could easily have shown that a 1-jet must satisfy equation (4.3) without ever considering the commutative diagrams. This will be the case in more complex examples as well — when considering our differential equation we shall be able to "spot" the obstructions and thus do without the the tedious business of actually considering the zig-zag map. The crucial ingredient in finding obstructions that is provided by the general theory is the knowledge that the obstructions all lie in W. Thus the dimension of W tells us "how many" obstructions there are. Thus in our example we know that there is an obstruction lying in $\bigwedge^2 \otimes E$, one guesses immediately that this is given by the curvature and so can write down an obstruction that lies in $\bigwedge^2$. We then know that we have written down all the obstructions.

In summary, if one finds the cokernel of the symbol (or its prolongation) this tells one how many obstructions there are and in which space they reside. This tells us where to look to find the obstructions and lets us know when we have found them all. Combining this with the observation that "all obstructions arise from the fact that derivatives commute" one can often write down the obstructions once one knows the cokernel of the symbol. At this point it may be of some practical assistance to notice that the Ricci identity and the fact that $\mathrm{d}^2 = 0$ are both expressions of the fact that derivatives commute.

One might feel that "guessing" the obstructions is pointless given that one has this explicit zig-zag map to calculate obstructions with. However, this latter approach is liable to produce lengthy formulae in local coordinates that one will then have to interpret. Guessing is often much the simpler alternative.

4.2.4 Cartan's test

The material in the previous section gives us one of the key steps in Cartan–Kähler theory. In particular we have a method of deciding whether or not l-jet solutions to a given differential equation can be extended to $(l+1)$-jet solutions. Thus one can, in principle, check whether or not a given differential equation admits solutions of any given order. Of course, we need some way of testing whether or not a given differential equation admits solutions of all orders. This is provided by what we shall call Cartan's test.

We have already shown that the question of whether or not one can extend l-jet solutions to a given differential equation depends upon algebraic properties of the symbol. An advanced algebraic theory has been devised to understand the symbol — in particular Spencer cohomology. In fact the obstructions we discussed in the previous section can be shown to lie in the Spencer cohomology groups of the symbol, and thus that a differential equation must be formally integrable if all the Spencer cohomology groups vanish. The interested reader should see [Gol67b]. However, one still requires a practical test to see if the Spencer cohomology vanishes. This is provided by a theorem due to J.-P. Serre [GS64]. Thus we have a practical test to see if a differential equation is formally integrable which we now state.

Definition 4.2.7 *Let us write g_{i+k} for the kernel of $\sigma_i(D)$, where D is a k-th order differential operator.*

If $\{e^1, \dots e^n\}$ is a basis for T^*X then write $S^l_{(j+1,\dots,n)}T^*$ for the subspace of S^lT^* generated by symmetric products of the $e^{j+1}, \dots, e^n$.

Define $g_{l,(j+1,\dots,n)} = g_l \cap S^l_{(j+1,\dots n))}T^*$.

Define $g_{l,j} = \dim g_{l,(j+1,\dots,n)}$.

A basis $(e^1, \dots, e^n)$ is said to be semi-regular *for g_l if:*

$$\dim g_{l+1} = \sum_{i=1}^{n} g_{l,i}.$$

g_l is said to be involutive *if there exists a semi-regular basis.*

Theorem 4.2.8 (Cartan's Test) *[Gol67b] [GS64] [Car45] If g_{k+i+1} has locally constant dimension, if $\pi : N_{k+i+1} \longrightarrow N_{k+i}$ is onto and $(g_{k+i})_e$ is involutive for all $e \in N_{k+i}$ then the equation $p_i(D) = f$ is formally integrable.*

Thus one sees that the equation will, in these circumstances, have solutions of any given order. Moreover, any solution of a given order can be extended indefinitely to a solution of any desired order. All one needs now is some estimate on the rate of growth of these solutions to obtain a genuine power series solution to the equation. Such an estimate is provided by the δ-Poincaré estimate, [Swe67], [EGS62]. One then has, [Gol67b], that an involutive differential equation admits analytic solutions — indeed a solution of any given order can always be extended to an analytic solution.

There are two remaining problems. Firstly, even if the differential equation is involutive, not all bases will be semi-regular for the given differential equation. However, it is easy to see that if one basis is semi-regular, then the generic basis will be semi-regular. Thus to see if a given differential equation is involutive one can just pick a random basis and test that basis.

Secondly, formal integrability does not imply involutivity. Thus even if our equation is formally integrable, our test may not observe this. The resolution of this problem is given by a theorem due to Kuranishi [Kur62] (see also [Qui64] for an algebraic counterpart to this result) — which we can interpret as saying that if we prolong a formally integrable differential equation sufficiently often, then we shall obtain an involutive differential equation. There are some technical provisos that one should make, however, in practice these do not seem to be important: one does not appeal to the result directly, one uses it as a moral justification for using the test for formal integrability that we have given even though it only tests for a sufficient condition.

In summary, we have a general strategy for finding out whether or not a given differential equation has solutions. First one computes the $g_{i,j}$ for the given differential equation. If we find that the equation is involutive and its first prolongation has surjective symbol then we are done. If it turns out that the equation does not have a surjective symbol then we find the obstructions, and consider the system obtained by appending the obstructions to the original differential equations. If the first prolongation has surjective symbol but is not involutive then one simply considers the first prolongation as a differential equation in its own right.

Thus one obtains a sequence of systems of differential equations which one hopes will end either with a system to which one can apply Cartan's test or with a system which has no solutions. The final ingredient one needs is a theorem due to Kuranishi [Kur62] which effectively states that, at least in practice this process will terminate eventually.

Of course, one need not apply Cartan's test explicitly. If one is lucky one might be able to to prolong the equation and calculate the effect of obstructions until one obtained an elliptic equation or an equation of Frobenius type. Nevertheless, the basic principle will be the same: one prolongs and considers the effect of the fact that derivatives commute to calculate obstructions until one reaches a point where one can appeal to some general theorem on the existence of solutions to differential equations.

In Chapter 3 we considered the differential equation given by the problem of finding out whether or not a given metric admits a compatible almost-Kähler structure. We showed how one could prolong the equations and calculate obstructions to reduce the problem to one of Frobenius type and thus answered the question "How can one tell if a given Riemannian manifold admits a compatible almost-Kähler structure?". Thus the "strategy" of Chapter 3 is nothing but Cartan–Kähler theory.

Notice then that Cartan–Kähler theory is often as useful as a guiding principle as much as an explicit tool. The lengthy calculations of Chapter 3 would have been impossible without such a guiding principle.

In the next section, we shall approach the question of whether or not there exist almost-Kähler Einstein manifolds and try to tackle it using Cartan–Kähler theory.

4.3 Applications to almost-Kähler, Einstein manifolds

Let $\mathcal{E}$ be the space of metrics compatible with the standard symplectic form on $\mathbb{R}^4$. The Ricci tensor defines a differential operator Ric mapping sections of $\mathcal{E}$ to sections of S^2T^*. Our aim in this section is to try to apply Cartan–Kähler theory to the differential equation $\operatorname{Ric} g = -\lambda g$ where λ is some constant.

If one chooses some $g \in \mathcal{E}$, we have that the vertical tangent space to g in $\mathcal{E}$ is isomorphic to the 6-dimensional space $E = [\![S^2(T^{1,0})]\!]$. We also need the fact that the symbol of the Ricci tensor at g is given by:

$$\sigma(\mathrm{Ric}) : S^2T^* \otimes S^2T^* \longrightarrow S^2T^*$$

by

$$\begin{aligned}&\sigma(\mathrm{Ric})(C)(\xi,\eta) = \\ &\quad -\frac{1}{2}\Sigma_{i=1}^{4}(C(e_i,e_i,\xi,\eta)+C(\xi,\eta,e_i,e_i)-C(e_i,\xi,e_i,\eta)-C(e_i,\eta,e_i,\xi)).\end{aligned} \tag{4.4}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis. The proof of this is a simple corollary of the standard formulae for the Riemann curvature tensor in terms of the metric and the Christoffel symbols.

We can now easily compute that we have the following exact sequence:

$$S^2T^* \otimes E@>\sigma(\mathrm{Ric})>> S^2T^* \longrightarrow 0.$$

This shows that there is certainly no difficulty in finding 2-jet solutions to the differential equation $\mathrm{Ric}\, g = \lambda g$, or indeed to the differential equation $\mathrm{Ric}\, g = \rho$ where ρ is some arbitary section of S^2T^*.

Now consider the prolongation:

$$S^3T^* \otimes E@>\sigma_1(\mathrm{Ric})>> T^* \otimes S^2T^*.$$

It is easy to check that $\sigma_1(\mathrm{Ric})$ is not onto and thus we should now look for some obstruction. We do not have to look far: the differential Bianchi identity tells us that $\beta(\nabla \mathrm{Ric}\, g) = 0$ where β is the SO(4) equivariant map from $T^* \otimes S^2T^* \longrightarrow T^*$ given by:

$$\beta(C)(\xi) = \sum (C(e_i, e_i, \xi) - \frac{1}{2} C(\xi, e_i, e_i)).$$

From the 1-jet of the metric and from a section ρ we can compute $\nabla\rho$. Thus a 2-jet solution of the equation $\mathrm{Ric}\, g = \rho$ only if its 1-jet satisfies $\beta(\nabla\rho) = 0$. However one easily sees that we have the exact sequence:

$$S^2T^* \otimes E@>\sigma_1(\mathrm{Ric})>> T^* \otimes S^2T^*@>\beta>> T^* \longrightarrow 0.$$

Thus we have found all the obstructions to extending 2-jet solutions of the differential equation $\mathrm{Ric}\, g = \rho$ to 3-jet solutions. We have also shown that the only obstruction to extending a 2-jet solution of the differential equation $\mathrm{Ric}\, g = \lambda g$ is that we must have $\beta(\nabla\lambda g) = 0$. Equivalently one must have that $\mathrm{d}\lambda = 0$. Thus the fact that any metric with $R_0 = 0$ necessarily has constant scalar curvature arises naturally as an obstruction in Cartan–Kähler theory.

In fact, if one considers the equation $\mathrm{Ric}\, g = \lambda g$ as a differential equation for a metric not necessarily compatible with the standard symplectic form, there are no further obstructions. This is now just an application of Cartan's test, [Gas82].

However, we are considering the problem of finding an Einstein metric compatible with the canonical symplectic form on $\mathbb{R}^4$. As we shall see in the next section there are further obstructions.

4.3.1 An obstruction to lifting 3-jet solutions to 4-jet solutions

We have the following exact sequences:

$$S^2T^* \otimes E@ > \sigma(\mathrm{Ric}) >> S^2T^* \longrightarrow 0,$$

$$S^3T^* \otimes E@ > \sigma_1(\mathrm{Ric}) >> T^* \otimes S^2T^*@ > \beta >> T^* \longrightarrow 0.$$

Thus we see that one can always extend a 2-jet solutions of the 4-dimensional almost-Kähler, Einstein equations to a 3-jet solution. However, the sequence:

$$S^4T^* \otimes E@ > \sigma_2(\mathrm{Ric}) >> S^2T^* \otimes S^2T^*@ > \sigma_1(\beta) >> T^* \otimes T^* \longrightarrow 0$$

is not exact. By calculating the dimension of the image of $\sigma_2(\mathrm{Ric})$ we see that there must in fact be some equivariant map $\gamma : S^2T^* \otimes S^2T^* \longrightarrow \mathbb{R}$ such that:

$$S^4T^* \otimes E@ > \sigma_2(\mathrm{Ric}) >> S^2T^* \otimes S^2T^*@ > \sigma_1(\beta) \oplus \gamma >> T^* \otimes T^* \oplus \mathbb{R} \longrightarrow 0$$

is exact. Thus there is some obstruction to extending 3-jet solutions of the almost-Kähler, Einstein equations. We wish to find out in more detail what this obstruction is.

Consider the SO(4) decomposition of $T^* \otimes T^* \otimes \bigwedge^2$.

$$T^* \otimes T^* \otimes \bigwedge^2 \cong (\overbrace{S^2V^+ \oplus S^2V^-}^{\bigwedge^2} \oplus \overbrace{S^2V^+ \otimes S^2V^- \oplus \mathbb{C}}^{S^2T^*}) \otimes (S^2V^+ \oplus S^2V^-)$$

$$\cong \begin{array}{l} \left.\begin{array}{l} S^4V^+ \oplus S^2V^+ \oplus \mathbb{C} \oplus S^2V^- \otimes S^2V^+ \end{array}\right\} \subseteq \bigwedge^2 \otimes S^2V^+ \\ \left.\begin{array}{l} \oplus S^4V^+ \otimes S^2V^- \oplus S^2V^+ \otimes S^2V^- \\ \oplus S^2V^- \oplus S^2V^+ \end{array}\right\} \subseteq S^2T^* \otimes S^2V^+ \\ \left.\begin{array}{l} \oplus S^2V^+ \otimes S^2V^- \oplus S^4V^- \oplus S^2V^- \oplus \mathbb{C} \end{array}\right\} \subseteq \bigwedge^2 \otimes S^2V^- \\ \left.\begin{array}{l} \oplus S^2V^+ \otimes S^4V^- \oplus S^2V^+ \otimes S^2V^- \\ \oplus S^2V^+ \oplus S^2V^- \end{array}\right\} \subseteq S^2T^* \otimes S^2V^- \end{array}$$

(the braces indicate where each term lies). So there are three S^2V^+ terms. Thus we can define three SO(4) equivariant maps τ^1, τ^2 and τ^3 taking $T^* \otimes T^* \otimes \bigwedge^2$ to the copies of S^2V^+ in $\bigwedge^2 \otimes S^2V^+$, $S^2T^* \otimes S^2V^+$ and $S^2T^* \otimes S^2V^-$ respectively.

Now let $b : T^* \otimes T^* \otimes \bigwedge^2 \longrightarrow T^* \otimes \bigwedge^3$ by $A_{ijkl} \longrightarrow A_{i[jkl]}$ and define $\mathcal{A}$ to be the kernel of b. Now

$$T^* \otimes \bigwedge^3 \cong S^2V^+ \otimes S^2V^- \oplus S^2V^+ \oplus S^2V^- \oplus \mathbb{C}$$

and b is clearly onto. So there exist constants α_1, α_2 and α_3 such that for all $A \in \mathcal{A}$

$$\alpha_1\tau^1(A) + \alpha_2\tau^2(A) + \alpha_3\tau^3(A) = 0.$$

$\omega \in S^2V^+$ so we have

$$\langle \omega, \alpha_1\tau^1(A) + \alpha_2\tau^2(A) + \alpha_3\tau^3(A)\rangle = 0.$$

Now $(\nabla\nabla R)_{abijkl}\omega^{kl} = \langle \nabla\nabla R, \omega\rangle \in \mathcal{A}$ by the differential Bianchi identity. So we shall apply this formula to $\langle \nabla\nabla R, \omega\rangle$.

By the Ricci identity, the components of $\nabla\nabla R$ in $\bigwedge^2 \otimes \bigwedge^2 \otimes \bigwedge^2$ are determined by $R \otimes R$. Since $\tau^1(A)$ is determined by the components of A in $\bigwedge^2 \otimes \bigwedge^2$, we can rewrite $\tau^1(\langle \nabla\nabla R, \omega\rangle)$ as $\rho(R \otimes R)$ for some U(2) equivariant mapping ρ from $\mathcal{R} \otimes \mathcal{R} \longrightarrow \mathbb{C}$.

By definition of τ^2, $\langle \omega, \tau^2(\langle R, \omega\rangle)\rangle = \omega^{ij}\omega^{kl}(\nabla\nabla R)_{aaijkl}$ (up to some constant). So we see that

$$\begin{aligned}
&\tau^2(\langle R, \omega\rangle) = \\
&\quad (\nabla\nabla(\omega^{ij}\omega^{kl}R_{ijkl}))_{aa} + \rho(\xi \otimes \overline{\nabla}R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R) \\
&\qquad = \Delta s^* + \rho(\xi \otimes \overline{\nabla}R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R)
\end{aligned}$$

(here ρ denotes a general U(2) equivariant map to $\mathbb{C}$, Δ is the Laplacian and we are ignoring constants).

If we denote by π^- the map $\mathcal{R} \longrightarrow \bigwedge^- \otimes \bigwedge^2$, then the fact that τ^3 depends only on the $T^* \otimes T^* \otimes S^2V^-$ term of A tells us that

$$\begin{aligned}
\tau^3(\langle \nabla\nabla R, \omega\rangle) &= \tau^3(\langle \pi^-\nabla\nabla R, \omega\rangle) = \tau^3(\langle \nabla\nabla\pi^- R, \omega\rangle) \\
&= \rho(\nabla\nabla(\pi^- R_{ijkl}\omega^{kl}) + \rho(\xi \otimes \overline{\nabla}R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R) \\
&= \rho(\nabla\nabla(R_F)) + \rho(\xi \otimes \overline{\nabla}R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R)
\end{aligned}$$

Putting all of this together we have:

$$\begin{aligned}
\rho(R \otimes R) + \rho(\xi \otimes \overline{\nabla}R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R) &= \Delta s^* + \rho(\nabla\nabla R_F) \\
= \Delta s + \Delta(s^* - s) + \rho(\nabla\nabla R_F) &= \Delta s + \Delta\|\xi\|^2 + \rho(\nabla\nabla R_F)
\end{aligned}$$

All of these functions except Δs and $\rho(\nabla\nabla R_F)$ are functions of the 3-jet. Δs and $\rho(\nabla\nabla R_F)$ can both be written as $\rho(\bigodot \nabla\nabla \,\mathrm{Ric})$.

Thus:

Proposition 4.3.1 *On an almost-Kähler* 4*-manifold:*

$$\rho(R \otimes R) + \rho(\xi \otimes \overline{\nabla} R) + \rho(\overline{\nabla}\xi \otimes R) + \rho(\xi \otimes \xi \otimes R) - \Delta\|\xi\|^2$$
$$= \rho(\bigodot \nabla\nabla \,\mathrm{Ric}) = \gamma(\bigodot \nabla\nabla \,\mathrm{Ric})$$

where the ρ's represent $\mathrm{U}(2)$ *equivariant maps to* $\mathbb{C}$.

We wish to understand this formula better. As a first step, we see what Schur's lemma can tell us about it.

From now on we assume that our 4-manifold is Einstein. Let us define a $\bigwedge^{2,0}$ valued $(1,0)$-form, η, by $\xi = \eta + \overline{\eta}$. We can then define operators ∂, ∂^* acting on $\bigwedge^{2,0}$ valued forms in the standard way. Note that W^+_{00} is proportional to $\partial\eta$ and W^+_F is proportional to $\partial^*\eta$.

Since each ρ is an equivariant map to $\mathbb{C}$, we can write:

$$\rho(R \otimes R) = c\|W^-\|^2 + c\|W^+_F\|^2 + c\|W^+_{00}\|^2 + ca^2 + cb^2 + cab$$

where c represents a general constant and a and b are the scalar components of R defined in Section 1.2.1.

Since $\rho(\xi \otimes \overline{\nabla} R) \in \mathbb{C}$, the only terms of $\overline{\nabla} R$ that effect $\rho(\xi \otimes \overline{\nabla} R)$ are those lying in spaces isomorphic to $[\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$. By looking at the relevant $\mathrm{U}(2)$ decompositions and by using Lemma 3.3.1, we see that:

$$\rho(\xi \otimes \overline{\nabla} R) = c(\partial\partial^*\eta, \eta) + c(\partial^*\partial\eta, \eta),$$

Similarly,

$$\rho(\overline{\nabla}\xi \otimes R) = c\|W^+_F\|^2 + c\|W^+_{00}\|^2,$$
$$\rho(\xi \otimes \xi \otimes R) = (ca + cb)\|\xi\|^2.$$

Using Maple to identify the constants, we get:

Theorem 4.3.2 *If (M^4, g, J) is almost-Kähler and Einstein then:*

$$6\Delta\|\xi\|^2 - 18\|W^+_F\|^2 + 12\|W^+_{00}\|^2 + 24(\partial\partial^*\eta, \eta) + 24(\overline{\partial\partial^*}\overline{\eta}, \overline{\eta}) = 3(2a - b)b. \tag{4.5}$$

A Maple program to calculate the constants is given in Appendix A.3.

4.3.2 An application to the compact case

Although equation (4.5) *looks* a lot like a Weitzenböck formula, it is not a Weitzenböck formula. Weitzenböck formulae arise in a completely different way and are not obstructions to extending solutions. There are Weitzenböck formulae on almost-Kähler manifolds that we shall discuss in the next section. The crucial difference is that one need only consider the 3-jet to prove the Weitzenböck formulae, whereas to prove our equation, which is still an equation on the 3-jet, one must consider the 4-jet.

Nevertheless, one may still hope to obtain information from our equation in much the same way as one obtains information from Weitzenböck formulae. One is tempted to integrate equation (4.5), however, the only information one gains is a new proof of Sekigawa's integral formula.

Nevertheless, we can extract some useful information. First we need:

Lemma 4.3.3 *Suppose that (M^4, g, J) is an almost-Kähler, Einstein manifold with $W^+_{00} \equiv 0$ then $(\partial\partial^*\eta, \eta) = c(2a-b)b$.*

Proof: *By our spinor formula for the curvature:*

$$W^+ = w^+_F u^3 \tilde{u} + \frac{3s^* - s}{8} u^2 \tilde{u}^2 - \overline{w^+_F} u \tilde{u}^3.$$

The differential Bianchi identity tells us that the component of ∇W^+ in $V^- \otimes S^3 V^+$ is equal to zero. Hence differentiating our formula for W^+ and looking at the component in $\langle u^3 \rangle \otimes V^-$, we see that:

$$M \otimes u^3 + c(3s^* - s)\tilde{\beta} u^3 = 0$$

where $\mathrm{d}w^+_F = M \otimes u + N \otimes \tilde{u}$. The result now follows since M is proportional to $\partial\partial^*\eta$, β is proportional to η, (η, η) is proportional to b and $3s^* - s$ is proportional to $2a - b$. $\square$

So, combining our results with the fact that $\|\xi\|^2 = -b$, we see that on an almost-Kähler, Einstein 4-manifold with $W^+_{00} \equiv 0$:

$$\Delta \|\xi\|^2 + \lambda \|\xi\|^2 \geq 0$$

where λ is some smooth (possibly positive) function.

Now recall the formulation of Hopf's maximum principle given in [Pro88],

Theorem 4.3.4 *Suppose that*

$$L = \sum a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial}{\partial x_i}$$

is a second order operator defined on a region $\Omega \subseteq \mathbb{R}^n$ and that L is uniformly elliptic — i.e. that there exists an η independent of x such that:

$$\sum a_{ij}(x)\xi_i\xi_j \geq \eta \sum \xi_i^2$$

for all vectors ξ. Then if a function u satisfies $Lu \geq 0$ in Ω, the a_{ij}, b_i are bounded in Ω and u attains a maximum, M, at $x \in \Omega$ then $u \equiv M$ in Ω.

Corollary 4.3.5 *Suppose (M, g) is a compact connected Riemannian manifold and $\Delta f + \lambda f \geq 0$ for some function f with $f \geq 0$ everywhere and some function $\lambda : M \longrightarrow \mathbb{R}$. Then if $f = 0$ at some point p then $f \equiv 0$.*

Proof: *Without loss of generality, $|\lambda| \leq 100$ everywhere. Choose geodesic coordinates centred at p. In these coordinates we can write*

$$\Delta + 100 = -\sum a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \sum b_i \frac{\partial}{\partial x_i} - c.$$

We can choose an h (defined in a suitably small neighbourhood of p) which satisfies $h(p) = 1$ and $\Delta h + 100h \equiv 0$. Now define g near p by $f = gh$. Then, near p, $g \geq 0$ and g attains a minimum at p.

Now $\Delta f = -h(Lg)$ where L is a uniformly elliptic operator with bounded a_{ij}'s and b_i's if we are near enough to p. Applying the maximum principle to $-g$, we see that $g \equiv 0$ in some neighbourhood of p.

Thus $f \equiv 0$. □

Theorem 4.3.6 *If (M^4, g, J) is a compact almost-Kähler, Einstein manifold with $W^+_{00} \equiv 0$ then it is necessarily Kähler.*

Proof: *It is proved in Theorem 2.2.4 that ξ must be equal to zero somewhere on a compact almost-Kähler, Einstein manifold. But $\Delta\|\xi\|^2 + \lambda\|\xi\|^2 \geq 0$. So ξ is identically zero.* □

Notice that the signs in equation (4.5) are crucial to the proof of this result. This is why we go to such lengths in Appendix A.3 to identify the constants correctly.

4.3.3 Prolonging again

Let us write $\mathcal{O}^1 : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathbb{R})$ for the map determined by this obstruction. That is to say:

$$\mathcal{O}^1 g = 6\Delta\|\xi\|^2 - 18\|W^+_F\|^2 + 12\|W^+_{00}\|^2 + 24(\partial\partial^*\eta, \eta) + 24(\overline{\partial\partial}^*\overline{\eta}, \overline{\eta})$$

So $\mathcal{O}^1$ is a third order differential operator. The strategy given to us by Cartan Kähler theory tells us that we must now consider the pair of third order equations:

$$\begin{aligned} p_1(\operatorname{Ric} g - \lambda g) &= 0 \\ \mathcal{O}^1 g &= 0 \end{aligned}$$

One has the exact sequences:

$$S^3T^* \otimes E@ > \sigma_1(\mathrm{Ric}) \oplus \sigma(\mathcal{O}^1) >> T^* \otimes S^2T^* \oplus R@ > \beta >> T^* \longrightarrow 0 \tag{4.6}$$

$$S^4T^* \otimes E@ > \sigma_2(\mathrm{Ric}) \oplus \sigma_1(\mathcal{O}^1) >> S^2T^* \otimes S^2T^* \oplus T^*@ > \sigma_1(\beta) \oplus \gamma >> T^* \otimes T^* \oplus \mathbb{R} \longrightarrow 0 \tag{4.7}$$

But one can compute that the next sequence is not exact: one needs to introduce a map

$$\delta : S^3T^* \oplus S^2T^* \oplus S^2T^* \longrightarrow \bigwedge\nolimits^- \oplus \mathbb{R}$$

representing the additional cokernel. One has the exact sequence:

$$S^5T^* \otimes E@ > \begin{array}{c} \sigma_3(\mathrm{Ric}) \\ \oplus\, \sigma_2(\mathcal{O}^1) \end{array} >> \begin{array}{c} S^3T^* \otimes S^2T^* \\ \oplus\, S^2T^* \end{array} @ > \begin{array}{c} \sigma_2(\beta) \\ \oplus\, \sigma_1(\gamma) \\ \oplus\, \delta \end{array} >> \begin{array}{c} S^2T^* \otimes T^* \\ \oplus\, T^* \\ \oplus\, \mathbb{R} \oplus \bigwedge^- \end{array} \longrightarrow 0.$$

Thus there is some obstruction to lifting 4-jet solutions of our problem to 5-jets which lies in $\bigwedge^- \oplus \mathbb{R}$.

Once again, we would like to understand this second obstruction a little better — after all if we are to continue to apply Cartan Kähler theory we at least need to know the symbol of the obstruction. We shall find a more explicit version of the obstruction in the next section. This will allow us to consider one further prolongation of the problem. However, it will turn out that there are yet more obstructions. On observing this, we shall stop analysing the problem any further. The reader should be warned before reading the next section that we do not have any application of our more explicit version of this second obstruction other than to prove that there is a third.

4.3.4 The second obstruction

We wish to outline how the second obstruction arises. We shall give sufficient detail that calculating the obstruction explicitly becomes, in principle, just a routine matter — a case of filling in the additional terms where we instead say "plus lower order terms". However, filling in the lower order terms would be a substantial task. The author found sufficient information about the obstruction to calculate its symbol, however, even doing this was time consuming and the author made heavy use of MAPLE.

We begin by defining three second order operators $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ each acting on sections of $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$.

Firstly, let us define a differential operator $\mathcal{D}_1$ by the formula $\mathcal{D}_1\eta = c\partial\partial^*\eta + c\partial^*\partial\eta$ where the c's are constants chosen so that we can write the formula of Proposition 4.3.1 as:

$$(\mathcal{D}_1\eta, \overline{\eta}) + \text{conjugate} = \rho(\nabla\nabla\,\mathrm{Ric}) + \text{lower order terms}. \tag{4.8}$$

where ρ is some U(2) equivariant map.

$\overline{\nabla}\eta$ has four U(2) components, two of which we have already called $\partial\eta$ and $\partial^*\eta$. Let us denote the component of $\overline{\nabla}\eta$ in $\langle u^4\rangle \otimes S^2V^-$ by $D_1\eta$ and the component in $\langle u^2\rangle \otimes S^2V^-$ by $D_2\eta$. Of course, $D_2\eta+$conjugate is essentially equal to R_{00}.

Now $\overline{\nabla\nabla}\eta \in T^* \otimes T^* \otimes \bigwedge^{1,0} \otimes \bigwedge^{2,0}$ has four components isomorphic to $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$. These are determined by $D_1^*D_1\eta$, $D_2^*D_2\eta$, $\partial^*\partial\eta$, $\partial\partial^*\eta$. However, $\bigwedge^2 \otimes \bigwedge^{1,0} \otimes \bigwedge^{2,0}$ has two components isomorphic to $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$. Thus, by the Ricci identity we must get two linearly independent Weitzenböck formulae of the form:

$$cD_1^*D_1\eta + cD_2^*D_2\eta + c\partial^*\partial\eta + c\partial\partial^*\eta = \text{lower order terms}$$

where the c's represent constants.

Once one identifies the constants, one can easily check that we may write the equations as:

$$\begin{aligned} D_1^*D_1\eta &= c\partial^*\partial\eta + c\partial\partial^*\eta + \text{lower order terms} \\ D_2^*D_2\eta &= c\partial^*\partial\eta + c\partial\partial^*\eta + \text{lower order terms} \end{aligned}$$

Let us define a differential operator $\mathcal{D}_2$ to be the linear combination of $\partial^*\partial\eta$ and $\partial\partial^*$ such that the second equation becomes:

$$D_2^* D_2 \eta = \mathcal{D}_2\eta + \text{lower order terms.}$$

Since $D_2\eta$ is essentially equal to R_{00} we have that:

$$\mathcal{D}_2\eta = \rho(\nabla \operatorname{Ric}) + \text{lower order terms.} \tag{4.9}$$

If one ignores lower order terms, the Weyl tensor is determined by $\partial\eta$ and $\partial^*\eta$. Since $\overline{\nabla}\partial\eta$ has no components in $\bigwedge^{1,0}$, we see that the components of ∇W^+ in spaces isomorphic to $\bigwedge^{1,0} \cong \langle u\rangle \otimes V^-$ are determined by the component of $\overline{\nabla}\partial^*\eta$ in $\langle u\rangle \otimes V^-$ and lower order terms. We shall refer to this component of $\overline{\nabla}\partial^*\eta$ as $\mathcal{D}_3\eta$.

The differential Bianchi identity tells us that the components of ∇W^+ in $S^3V^+ \otimes V^-$ are determined by $\nabla \operatorname{Ric}$. Thus we have that:

$$\mathcal{D}_3\eta = \rho(\nabla \operatorname{Ric}) + \text{lower order terms.} \tag{4.10}$$

The basic source of our obstruction is equation (4.8). If we differentiate this equation twice and symmetrise we get an equation with components in $T^* \otimes T^*$. We are interested in particular in the components in $[S^2V^-]$ and $\mathbb{R}$.

We have that:

$$\rho(\nabla\nabla\nabla\nabla \operatorname{Ric}) = (\odot\overline{\nabla\nabla}\mathcal{D}_1\eta, \overline{\eta}) + \text{conjugate} + \text{lower order terms} \in S^2T^*.$$

Since the space of maps of $\bigwedge^{0,1} \otimes \bigwedge^{0,2}$ to $\mathbb{C}$ is isomorphic to $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$, the components of the above equation in $\mathbb{R}$ are determined entirely by the $\bigwedge^{1,0} \otimes \bigwedge^{2,0} \cong \langle u^3\rangle \otimes V^-$ components of $\odot\overline{\nabla\nabla}\mathcal{D}_1$ and lower order terms. Similarly since the space of maps of $\bigwedge^{0,1} \otimes \bigwedge^{0,2}$ to S^2V^- is isomorphic to $\langle u^3\rangle \otimes V^- \oplus \langle u^3\rangle \otimes S^2V^-$, the componnents of the above equation in $[S^2V^-]$ are determined by the $\langle u^3\rangle \otimes V^-$ and the $\langle u^3\rangle \otimes S^2V^-$ terms of $\odot\overline{\nabla\nabla}\mathcal{D}_1\eta$ and lower order terms.

Thus if we can express the components of $\overline{\nabla\nabla\nabla\nabla}\eta$ in $\langle u^3\rangle \otimes V^-$ and $\langle u^3\rangle \otimes S^2V^-$ in terms of the fourth jet we shall have an equation of the form:

$$\rho(\nabla\nabla\nabla\nabla \operatorname{Ric}) = \text{terms involving only the fourth jet}$$

which lies in $\mathbb{R} \oplus [S^2V^-]$. Thus we would have four real conditions of the form:

$$\text{terms involving only the fourth jet} = 0$$

which would have to hold on any almost-Kähler, Einstein four-manifold.

Now consider the decomposition:

$$\begin{aligned} S^4T^* \otimes \langle u^3 \rangle \otimes V^- \quad \cong \quad & (\langle u^7 \rangle \oplus \langle u^5 \rangle \oplus \langle u^3 \rangle \oplus \langle u \rangle \oplus \langle \tilde{u} \rangle) \otimes S^5V^- \\ & \oplus (\langle u^7 \oplus \rangle \oplus 2\langle u^5 \rangle \oplus 2\langle u^3 \rangle \oplus 2\langle u \rangle \oplus \langle \tilde{u} \rangle) \otimes S^3V^- \\ & \oplus (\langle u^5 \rangle \oplus \langle u^3 \rangle \oplus \langle u \rangle) \otimes V^- \end{aligned}$$

So by the Ricci identity, $\overline{\nabla\nabla\nabla\nabla}\eta$ has exactly two independent components in each of $\langle u^3 \rangle \otimes V^-$ and $\langle u^3 \rangle \otimes S^2V^-$ which are not determined by terms involving just the fourth jet. However, $\odot\overline{\nabla\nabla}\mathcal{D}_2\eta$ and $\odot\overline{\nabla\nabla}\mathcal{D}_3\eta$ have components lieing in these spaces. Thus by Schur's lemma and equations (4.9) and (4.10), the components of $\overline{\nabla\nabla\nabla\nabla}\eta$ in $\langle u^3 \rangle \otimes V^-$ and $\langle u^3 \rangle \otimes S^3V^-$ are determined by $\nabla\nabla\nabla\,\mathrm{Ric}$ and lower order terms.

Putting this together, we have identified four real equations on the 4-jet which must be satisfied by any almost-Kähler, Einstein 4-manifold. Let define a differential operator $\mathcal{O}_2$ mapping 4-jets to $\mathbb{R} \oplus [S^2V^-]$ representing these equations. One can calculate its symbol and then, using a computer, check that its symbol is linearly independent of $\sigma_2(\mathrm{Ric}) \oplus \sigma_1(\mathcal{O}_1)$. This tells us that our equations really do imply conditions that a four-jet must satisfy in order to be extended. Thus $\mathcal{O}_2$ represents the obstruction to extending 4-jet solutions predicted in the previous section.

Thus we now have the differential equations:

$$\begin{aligned} \nabla\nabla\,\mathrm{Ric}\,g &= 0 \\ \nabla\mathcal{O}_1 g &= 0 \\ \mathcal{O}_2 g &= 0 \end{aligned}$$

which any 4-jet of an almost-Kähler, Einstein metric must satisfy. One can now check that the symbol of the first prolongation of these equations does not have a surjective symbol using MAPLE. We see that there must be at least one further obstruction.

4.4 Hermitian Einstein manifolds

4.4.1 The Riemannian Goldberg–Sachs theorem

Clearly, one can apply Cartan–Kähler theory to the problem of finding Hermitian, Einstein manifolds. This time, let $\mathcal{E}$ be the space of metrics compatible with the standard almost-complex structure J on $\mathbb{R}^4$. Thus at a point $g \in \mathcal{E}$, the vertical tangent space of $\mathcal{E}$ is isomorphic to $\bigwedge_0^{1,1}$. One easily checks that we have the exact sequences:

$$S^2T^* \otimes E@ > \sigma(\mathrm{Ric}) >> S^2T^* \longrightarrow 0,$$

$$S^3T^* \otimes E@ > \sigma_1(\mathrm{Ric}) >> T^* \otimes S^2T^*@ > \beta >> T^* \longrightarrow 0.$$

More interestingly, the next sequence fails to be exact — there is a 1-dimensional cokernel:

$$S^4T^* \otimes E@ > \sigma_2(\mathrm{Ric}) >> S^2T^* \otimes S^2T^*@ > \sigma_1(\beta) >> T^* \otimes T^* \longrightarrow 0$$

Thus there is some obstruction to the possibility of extending 3-jet solutions of the problem to 4-jet solutions which needs to be found. In fact, the obstruction is the Riemannian Goldberg–Sachs theorem:

Theorem 4.4.1 *[PB83] [Nur] [PR86b] [AG97] A Hermitian, Einstein* 4-*manifold necessarily satisfies* $W_F^+ = 0$.

Thus we have a condition that the curvature of any Hermitian Einstein manifold must obey and yet which one must examine the 4-jet of the metric to prove. Standard proofs of the Riemannian Goldberg–Sachs theorem do not mention the above sequences, but the author believes that the sequences give a useful insight into what causes the Riemannian Goldberg–Sachs theorem to arise. For example, the consideration of the fourth jet that one sees in all the proofs is essential. Futhermore, if we had not already known the Riemannian Goldberg–Sachs theorem, our sequences would tell us that there must be some such result that can be proved.

This is exactly what we have done in the previous section. Guided by the exact sequences Cartan–Kähler theory provides we have been lead to an equation 4.5 which one would not have otherwise had much reason to suspect.

The Riemannian Goldberg–Sachs theorem is known to have a number of remarkable corollaries. First notice that the Weyl tensor of a Hermitian Einstein manifold must be of the form:

$$W^+ = \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & b & 0 \\ 0 & 0 & b \end{array}\right).$$

The case when the manifold is neither self-dual or Kähler will henceforth be referred to as the case of a strictly HE manifold. In this case, W^+ has exactly 2 distinct eigenvalues. Einstein manifolds with such so-called algebraically special Weyl tensors have been studied by Derdzinski in [Der83]. He proves that the conformally related metric $2(3)^{1/3}|W^+|^{2/3}g$ is a so called *extremal Kähler* metric ([Cal82]) with respect to the complex structure given by the non-repeated eigenvector. In the study of Hermitian Einstein manifolds, factors such as $2(3)^{1/3}|W^+|^{2/3}$ will occur often. The details of finding the correct factors will be easy. Thus in what follows we shall describe various facts about Hermitian Einstein manifolds "up to a factor involving $|W^+|$" so as to minimize the need for explicit calculation.

This tells us immediately, [Cal82], that the rescaled metric admits a non-vanishing Killing vector field, and thus so does the original metric. It is easy to see that, up to a factor involving $|W^+|$, this Killing vector field is essentially equal to the Lee form. Although we have used results of [Cal82] and [Der83], the fact that the Lee form can be rescaled to give a Killing vector field can be easily proved using results analagous to Theorem 1.2.2 and those of Section 3.3.2. If we assume now that our manifold is compact, Bochner's theorem tells us now that the scalar curvature is positive. The formulae for the first Chern class analagous to those we derived in Section 2.2.4 combined with the fact that $W_F^+ = 0$ quickly tell us that the anti-canonical line bundle of our manifold is ample, [GM94]. One can now combine this information with the classification of complex surfaces to show:

Theorem 4.4.2 *[LeB97] Let (M^4, g, J) be compact and admit a Hermitian, but non-Kähler, Einstein metric then as a complex manifold, M is obtained by blowing up $\mathbb{C}P^2$ at one, two or three points in general position.*

The existence of this Killing vector field is evidence of the remarkable power of the Riemannian Goldberg–Sachs theorem. Thus we wish to exhibit the Riemannian Goldberg–Sachs theorem as an example of the power of considering obstructions.

4.4.2 The SU(∞) Toda field equation

It is natural, therefore, to ask what further obstructions there are to the existence of strictly HE metrics. We shall answer this question by writing the equations for Hermitian Einstein metrics in a simple form.

Since a strictly HE manifold is conformally equivalent to a Kähler metric, we must have a moment map z for the Killing vector field. Indeed by the results of [Cal82] and [Der83], this moment map must be proportional to an appropriate power of $|W^+|$.

Let us take coordinates x, y, z and t exactly as in section 1.3.2. Then we can write the Hermitian, Einstein metric as:

$$g := W(e^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2) + \frac{1}{W}(\mathrm{d}t + \theta)^2.$$

We know that the self-dual Weyl tensor is determined by some (easily calculated) function of z. Thus if the metric is rescaled by this appropriate function of z we shall obtain a Kähler metric. Thus, just as the condition that the metric is Kähler allows one to determine $\mathrm{d}\theta$ in section 1.3.2, we can use this information to determine $\mathrm{d}\theta$ in this situation. The end result is:

$$\mathrm{d}\theta = \frac{W_z + W u_z - 2\frac{W}{z}}{W}\mathrm{d}x \wedge \mathrm{d}y - \frac{W_y}{We^{\frac{u}{2}}}\mathrm{d}x \wedge \mathrm{d}z + \frac{W_x}{We^{\frac{u}{2}}}\mathrm{d}y \wedge \mathrm{d}z.$$

Now just as in Section 1.3.2, we know the values we need to ascribe to the self-dual Weyl tensor and to the Ricci tensor. The self-dual Weyl tensor is determined by the fact that $W^+_F = 0$, $W^+_{00} = 0$ and the fact that $|W^+|$ is some function of z. The Ricci tensor should, of course, just be a multiple, Λ of the metric. In Section 1.3.2, knowing the self-dual Weyl tensor and the Ricci tensor was seen to determine W as a function of u and impose second order differential equation on u. We would certainly appear to have enough information to do the same kind of thing here. The details are a reasonably simple calculation. The end result is:

$$W = \frac{-6z(2 - zu_z)}{2\Lambda z^3 - 1},$$

$$u_{xx} + u_{yy} + e^u\left(u_{zz} + u_z^2 + \frac{12\Lambda(u_z z^2 + 2z)}{1 - 2\Lambda z^3}\right) = 0.$$

Finally one needs to check that if W, u and $\mathrm{d}\theta$ satisfy these equations, the integrability condition $\mathrm{d}\mathrm{d}\theta = 0$ automatically holds. This is indeed the case.

Theorem 4.4.3 *Strictly Hermitian Einstein 4-manifolds with scalar curvature Λ are determined by solutions to the equation.*

$$u_{xx} + u_{yy} + e^u \left(u_{zz} + u_z^2 + \frac{12\Lambda(u_z z^2 + 2z)}{1 - 2\Lambda z^3} \right) = 0. \tag{4.11}$$

When $\Lambda = 0$ this equation simplifies to the so-called $\mathrm{SU}(\infty)$*-Toda field equation. Thus one has a correspondence between Ricci flat Hermitian manifolds and anti-self-dual, Einstein manifolds with a Killing vector field.*

The zero scalar curvature case of the above result was proved by Przanowski and Bialecki in [PB87]. However, their proof requires use of Lie–Bäcklund transformations. In effect our choice of coordinate system provides a geometric interpretation of Przanowski and Bialecki's Lie–Bäcklund transformations in terms of moment maps.

Thus, in the $\Lambda = 0$, case we have found a differential equation which is clearly formally integrable which determines strictly HE four manifolds. Indeed the $\mathrm{SU}(\infty)$-Toda field equation forms an integrable system in the sense of [MW96]. In the $\Lambda \neq 0$ case, the differential equation still has the same symbol, and so it too must be formally integrable. Thus there are no further obstructions to the existence of 4-dimensional strictly HE manifolds.

Appendix A

The MAPLE programs required for Chapter 4

A.1 A procedure to perform the Einstein summation convention

Throughout this appendix we shall use a MAPLE procedure called `einstein` which allows one to use the notation of the Einstein summation convention in other MAPLE programs. The author has found this procedure to be useful in a number of contexts. In Section A.3 we shall use the procedure to compute the coefficients in equation (4.5). In Section A.2 we shall see how the procedure assists with computing the symbol of differential operators.

To use the procedure, a tensor named `name` should have its values stored as an array named `ename`. For example we may wish to set up a $(1, 1)$-tensor called `A` whose values might be stored as follows:

```
eA:=array(1..4,1..4,[
[1,2,3,4],
[5,6,7,8],
[4,3,2,1],
[8,7,6,5]]):
```

we may also have a vector called `B` with values determined by

```
eB:=array(1..4,[[1,2,3,4]]):
```

We wish to compute $\mathtt{C}^i = \mathtt{A}^i{}_j\mathtt{B}^j$ and store its values in `eC`. This is accomplished by typing:

```
einstein(C[i]=A[i,j]*B[j]):

eval(eC);
```

$$[30, 70, 20, 60]$$

Hopefully this makes the operation of `einstein` clear. One is allowed to use much more complex expressions than the one above — one can perform as many tensor products, sums of tensors and contractions as one likes. Also the tensor need not be a tensor of type (p, q), the only condition is that the array dimensions match in the indices one is summing over. Notice that the procedure does not distinguish between upper and lower indices.

It is hoped that the reader will readily believe in the existence of such a program and so we shall not include the actual code of the program as it is rather lengthy.

Notice that the order on multiplies tensors in makes an enormous difference to how long it takes to compute a given tensor. The procedure `einstein` contains an algorithm to choose a "sensible" order. For this reason it is not advisable to introduce too many tensors that are not directly involved in the computation. For example if one has tensors b, $T1$, $T2$, $T3$ and T and one wish to compute:

$$T1_{Bal}b_{Aij}T_{ijkl}b_{Bka} + T2_{Bal}b_{Aij}T_{ijkl}b_{Bka} + T3_{Bal}b_{Aij}T_{ijkl}b_{Bka}$$

by first introducing a tensor $S_{ABal} = b_{Aij}T_{ijkl}b_{Bka}$ and then computing $T1_{Bal}S_{ABal} + T2_{Bal}S_{ABal} + T3BalS_{ABal}$. One might be tempted to do this using the program:

```
einstein(S[A,B,a,l]=b[A,i,j]*T[i,j,k,l]*b[B,k,a]):
einstein(answer[A]=T1[B,a,l]*S[A,B,a,l] + T2[B,a,l]*S[A,B,a,l]
    + T3[B,a,l]*S[A,B,a,l]):
```

However, this might force the computer to multiply the tensors in an inefficient order. Instead should introduce the new tensor more symbolically. Thus one should write:

```
term:= R[i,j,k,l]*b[A,i,j]*b[B,k,a]:
einstein(answer[A]=T1[B,a,l]*term + T2[B,a,l]*term +
T3[B,a,l]*term):
```

Notice that if the input to `einstein` contains variables which have values ascribed to them as `term` does in the previous example, the procedure expands the input fully before performing the computation. As the above example shows, this is often convenient. However, if one temporarily ascribes a value to a variable such as `i` which often occurs as a formal index, one should take care to write "`i:='i':`" when one has finished using the variable. Indeed it is always good practice in MAPLE to unevaluate a variable when one has finished with it.

A.2 A program to assist with Cartan–Kähler theory

We wish to be able to compute the numbers $g_{i,j}$ from the symbol of the differential operator. The relevant definitions have already been given in Chapter 4, but we wish to write a computer program that can do the calculations for us.

Let E be a $\dim E$ dimensional vector bundle and let F be a $\dim F$ dimensional vector bundle over an n-dimensional Manifold.

Let $D : \Gamma(E) \longrightarrow \Gamma(F)$ be a differential operator of order d whose symbol is $\sigma(D)$. If we choose a basis $e^1, e^2, \ldots$ for E^* and a basis $f_1, f_2, \ldots$ for F and a basis $x_1, x_2, \ldots, x_n$ for TM then we can define `sigma` by:

$$\sigma(D) = \texttt{sigma[a,b,}\ldots\texttt{,c,u,v]}\, x_a \otimes x_b \ldots x_c \otimes E^u \otimes F_v.$$

This $(n \times n \times \ldots \times n \times \dim E \times \dim F)$ array will provide the basic input for our procedure.

Now $\sigma_i : S^{d+i}T^* \otimes E \longrightarrow S^iT^* \otimes F$ can easily be computed if one views it as an element of $T \otimes \ldots T \otimes E^* \otimes T^* \otimes \ldots T^* \otimes F$. However, in order to compute $\dim g_{d+i}$ we need to think of it as an

$$\dim(S^{d+i}T^* \otimes E) \times \dim(S^iT^* \otimes F)$$

array. To do this explicitly we need to pick bases for each of the two spaces. As a first step to doing this we pick a basis for S^iT.

One way to specify such a basis is to define an array `basis` which is

$$\binom{i+n-1}{i} \times \underbrace{n \times n \times \ldots \times n}_{i \text{ times}}$$

dimensional and such that

$$\texttt{Basis[alpha,a,b,}\ldots\texttt{,c]}\, u_a \otimes u_b \ldots u_c$$

runs through a basis for S^iT^* as `alpha` runs from 1 through to $\binom{i+n-1}{i}$.

Rather than compute such an array it is convenient to work with a somewhat smaller array `D`. `D` is an $n \times n \times \ldots \times n$ array from which one defines `basis` via:

$$\texttt{basis[alpha,a,b}\ldots\texttt{,c]} = \begin{cases} 1 & \text{if } \texttt{D[a,b,}\ldots\texttt{c]} = \texttt{alpha} \\ 0 & \text{otherwise} \end{cases}$$

We shall define `D` and hence `basis` by writing a MAPLE procedure which computes it. As examples though,

$$\mathtt{D} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \text{ when } i = 1 \text{ and } n = 4,$$

$$\mathtt{D} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix} \text{ when } i = 2 \text{ and } n = 4.$$

Before writing down the procedure that computes `D` we first need two subprocedures, both of which are a straightforward.

Firstly `computelist` which takes as input a number `i` and which outputs an $(\mathtt{i}!) \times \mathtt{i}$ array containing all the permutations of $1, 2, \ldots, \mathtt{i}$. For example:

$$\mathtt{computelist(3)} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Here is the procedure:

```
computelist:=proc(i)
local answer,counter2,a,j,term,counter3,flag3,flag2,t,count:
answer:=array(1..factorial(i),1..i):

A.0:=: counter2:= array(1..i):
for a from 1 to i do: A.0:=A.0 union a: counter2[a]:=a: od:
flag2:=0:
count:=1:

while not(flag2=1) do:

term:=NULL:
for j from 1 to i do:
answer[count,j]:=op(i+1-counter2[j],A.(j-1)):
A.j:=A.(j-1) minus answer[count,j]:
od:

counter3:=i:
flag3:=0:
while not(flag3=1) do:
counter2[counter3]:=counter2[counter3]+1: flag3:=1:
```

```
if counter2[counter3] > i then flag3:=0:
    counter2[counter3]:=counter3: counter3:=counter3-1: fi:
if flag3=0 and counter3 = 0 then flag3:=1: flag2:=1: fi:
od:

count:=count+1:
od:

evalm(answer): end:
```

The second procedure that we need, `increment`, takes as input a vector `counter` of any length and a number `n`. The procedure defines a new vector `counter1` by adding adds 1 to the final entry of `counter`. If this now exceeds `n`, the final entry is set to 1 and the penultimate entry is increased otherwise one stops. If no overflow occurs the output is:

`[counter1,0]`

if an overflow has occurred the output is:

`[[n,n,...n],1]`

For example,

```
increment([1,2,3,2],4)  =  [[1,2,3,3],0],
increment([1,2,4,4],4)  =  [[1,3,1,1],0],
increment([4,4,4,4],4)  =  [[4,4,4,4],1].
```

Here is the procedure:

```
increment:=proc(counter,n)
local flag1,length,counter1,flag2,counter2,a,i:
flag1:=0:
length:=nops(op(3,op(1,counter))):
counter1:=copy(counter):
if counter1[length]<n then counter1[length]:=
    counter1[length]+1:
else if counter1[1]=n then flag1:=1:
else flag2:=0: counter2:=length: while(flag2=0) do:
counter2:=counter2-1:
if not(counter1[counter2]=n) then flag2:=1 fi:
od:
a:=counter1[counter2]+1:
for i from counter2 to length do: counter1[i]:=a: od:
```

```
fi:
fi:
eval([eval(counter1),flag1]):
end:
```

With these preliminaries we can now define `D` to be the output of the following procedure when it is given the inputs i and n.

```
computeD:=proc(i,n)
local seq1,j,D,counter,list1,flag,count,a,temp,sequence:
seq1:=NULL:
for j from 1 to i do:
seq1:=seq1,1..n:
od:
D:=array(sparse,seq1):
counter:=array(1..i):
for j from 1 to i do:
counter[j]:=1:
od:
list1:=computelist(i):
flag:=0: count:=1:
while flag=0 do:
for a from 1 to factorial(i) do:
sequence:=seq(counter[list1[a,`j`]],`j`=1..i):
D[sequence]:=count:
od:
count:=count+1:
temp:=increment(counter,n):
counter:=copy(op(1,temp)): flag:=op(2,temp):
od:
eval(D):
end:
```

Given our basis for S^iT we can write the program which computes $\sigma_i(D)$ when viewed as a

$$\left(\binom{i+d-1+n}{i+d}(\dim E) \times \binom{i-1+n}{n}(\dim F)\right)$$

matrix.

To be precise we need to choose bases with respect to which we shall write σ_i. If we write $b^1, b^2, \ldots$ for our basis for $S^{i+d}T^*$ then we have a basis $b^1 \otimes e^1, b^1 \otimes e^2, \ldots b^2 \otimes e^1 \ldots b^2 \odot e^2, \ldots$ for $S^{i+d}T^* \otimes E$. We take the analogous basis for $S^iT^* \otimes F$.

The next procedure takes as input `sigma`, `dimE`, `dimF`, `n`, the dimension of the manifold, and `i`. It then outputs the matrix of $\sigma_{\mathtt{i}}(D)$ with respect to these bases.

```
prolongation:=proc(sigma,d,dimE,dimF,i,n)
local D,list1,answer,flag,counter,j,alpha,seq1,seq2,
    beta,z,count,a,temp:
if i=0 then answer:=specialcase(sigma,d,dimE,dimF,i,n):
else
D:=copy(computeD(i,n)):
list1:=copy(computelist(i+d)):
answer:=array(sparse,
    1..binomial(i+d+n-1,i+d)*dimE,1..binomial(i+n-1,i)*dimF):
flag:=0:
counter:=array(1..i+d):
for j from 1 to i+d do: counter[j]:=1: od:
count:=1:

while flag=0 do:
for alpha from 1 to dimE do:
for a from 1 to factorial(i+d) do:
seq1:=seq(counter[list1[a,`j`]],`j`=1..i): z:=D[seq1]:
seq2:=seq(counter[list1[a,`j`]],`j`=i+1..i+d):
for beta from 1 to dimF do:
answer[count,(z-1)*dimF + beta]:=answer[count,(z-1)*dimF+beta]
    + 1/factorial(i+d)*sigma[seq2,alpha,beta]:
od:
od:
count:=count+1:
od:
temp:=increment(counter,n):
counter:=copy(op(1,temp)): flag:=op(2,temp):
od:
fi:
evalm(answer):
end:

specialcase:=proc(sigma,d,dimE,dimF,i,n)
local D,list1,answer,flag,counter,j,alpha,seq1,seq2,
    beta,z,count,a,temp:

list1:=copy(computelist(d)):
answer:=array(sparse,1..binomial(d+n-1,d)*dimE,1..dimF):
flag:=0:
counter:=array(1..d):
for j from 1 to i+d do: counter[j]:=1: od:
count:=1:

while flag=0 do:
```

```
for alpha from 1 to dimE do:
for a from 1 to factorial(d) do:
seq2:=seq(counter[list1[a,`j`]],`j`=i+1..i+d):
for beta from 1 to dimF do:
answer[count,beta]:=answer[count,beta]
    + 1/factorial(i+d)*sigma[seq2,alpha,beta]:
od:
od:
count:=count+1:
od:
temp:=increment(counter,n):
counter:=copy(op(1,temp)): flag:=op(2,temp):
od:
evalm(answer):
end:
```

So given the datum `sigma` one can easily compute $\dim g_{i+d}$. Indeed if one wishes to compute $g_{i+d,j}$ one need only compute the nullity of the matrix output by:

`prolongation(sigma,d,dimE,dimF,i,j).`

Notice that $x_1, x_2, \ldots, x_n$ determines the basis with respect to which we are computing $g_{i+d,j}$.

To see an illustration of the procedure `prolongation` we consider the equations for a four dimensional metric to be Einstein. We take $\mathcal{E}$ to be the space of metrics and so $E = S^2T^*$. We have $\mathcal{F} = F = S^2T^*$. We have the differential operator

$$\mathrm{Ric} - \lambda g : \Gamma(\mathcal{E}) \longrightarrow \Gamma(F)$$

which sends a metric to its Ricci tensor minus λg. The symbol of this $\sigma(\mathrm{Ric}) : S^2T^* \otimes S^2T^* \longrightarrow S^2T^*$ is given by equation (4.4).

We wish to compute `sigma` from this formula. With our procedure `einstein` this is easy. First we choose a basis for $S^2T^* = E = F$, we shall call this `bS2` it is given as follows:

```
ebS2:=array(1..10,1..4,1..4,[
[[1,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]],
[[0,0,0,0],[0,1,0,0],[0,0,0,0],[0,0,0,0]],
[[0,0,0,0],[0,0,0,0],[0,0,1,0],[0,0,0,0]],
[[0,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,1]],
[[0,1,0,0],[1,0,0,0],[0,0,0,0],[0,0,0,0]],
[[0,0,1,0],[0,0,0,0],[1,0,0,0],[0,0,0,0]],
[[0,0,0,1],[0,0,0,0],[0,0,0,0],[1,0,0,0]],
[[0,0,0,0],[0,0,1,0],[0,1,0,0],[0,0,0,0]],
[[0,0,0,0],[0,0,0,1],[0,0,0,0],[0,1,0,0]],
```

```
[[0,0,0,0],[0,0,0,0],[0,0,0,1],[0,0,1,0]]]):
```

We also need a matrix containing the Kronecker delta:

```
eid:=array(identity,1..4,1..4):
```

With these preliminaries, we can compute the symbol of our differential operator by:

```
einstein(sigmaR[a,b,alpha,beta]=
 id[a,b]*id[c,i]*id[d,j]*bS2[alpha,c,d]*bS2[beta,i,j]
+id[c,d]*id[a,i]*id[b,j]*bS2[alpha,c,d]*bS2[beta,i,j]
-id[a,d]*id[b,j]*id[c,i]*bS2[alpha,c,d]*bS2[beta,i,j]
-id[b,c]*id[a,i]*id[d,j]*bS2[alpha,c,d]*bS2[beta,i,j]):
```

We now write a short procedure that will interpret the output of the procedure `prolongation` for us. It takes as input the symbol, the dimensions of E and F, the degree of the operator, the level i to which one wishes to prolong and the dimension of the manifold. It outputs pairs of numbers which are the nullity and the rank of σ_i as restricted to the span of $x_1, x_2 \dots x_a$ in $S^{d+i}T^*$. In particular the numbers on the left give the Cartan characters and the last number on the right tells us the rank of σ_i.

```
cartan:=proc(sigma,d,dimE,dimF,i,dimM)
local n,A,dimdomain,nullity,r:
for n from 1 to dimM do:
A:=prolongation(sigma,d,dimE,dimF,i,n):
dimdomain:=binomial(i+d+n-1,i+d)*dimE:
r:=rank(A):
nullity:=dimdomain-r:
print([nullity,r]);
od:
NULL:
end:
```

We now get the pay off:

```
cartan(esigmaR,2,10,10,0,4);
```

$$[4, 6]$$

$$[20, 10]$$

$$[50, 10]$$

$$[90, 10]$$

```
cartan(esigmaR,2,10,10,0,4);
```

$$[4, 6]$$

$$[24, 16]$$

$$[74, 26]$$

$$[164, 36]$$

So $g_{2,1} = 4$, $g_{2,2} = 20$, $g_{2,3} = 40$, $g_{2,4} = 90$ and $\dim g_3 = 164$. Also $\dim(\operatorname{Im}\sigma_1) = 36$). Thus g_2 is involutive but σ_1 is not onto: $\dim T^* \otimes S^2T^* = 40$.

A.3 The calculations required to prove equation (4.5)

Of course, we already have a good understanding of equation 4.5, all we wish to do is identify the constants. The proof of this is just a calculation — however, as we shall see it is a rather lengthy one. Although the calculation is long, it is very straightforward.

Our basic idea is to gradually build up an explicit example of the first few jets of an almost-Kähler, Einstein manifold. One can then use this explicit example to check formulae. The program was in fact built up over a long period of time as the author examined higher jets. Thus the specific calculations we perform do not necessarily provide the most efficient route to proving the equation. However, since the author has built up the following program over time and used it to check and reprove many results in almost-Kähler geometry one is reluctant to start from the beginning again.

This section provides an example of how our program `einstein` can be used to perform local calculations for us. The advantages of using a computer to assist in performing this calculation are clear: although one could in principle perform the calculation by hand, the signs in the formulae are crucial and so it is doubtful if such a calculation would be convincing.

There are a couple of differences between the notation used in this section and that used in the paper: we use the opposite sign convention for the Laplacian and the components of R called a and b have opposite signs to those used in the paper.

We begin by loading up the procedures that we shall need:

```
restart:
with(linalg):
read `einstein`:
```

A.3.1 Finding ξ and $\overline{\nabla}\xi$ compatible with R

The first thing we wish to do is set-up algebraic tensors R, $\overline{R}$, ξ and $\overline{\nabla}\xi$ which lie in the appropriate spaces and which are mutually compatible in so far as they satisfy the following:

$$\overline{R}_{X,Y}Z = R_{X,Y}Z + (\overline{\nabla}_{[X}\xi)_{Y]}Z + \xi_{\xi_X Y - \xi_Y X}Z - \xi_{[X}\xi_{Y]}Z$$

As we know, this tells us that three components of $\overline{\nabla}\xi$ are determined by R as is the norm of ξ when the manifold is almost-Kähler. We wish to write a computer program which takes as input an algebraic curvature tensor R and outputs tensors ξ, $\overline{\nabla}\xi$ and $\overline{R}$ compatible with the above equation, and with $\xi \in [\![A]\!]$, $\overline{\nabla}\xi \in T^*M \otimes [\![A]\!]$ ($A = \bigwedge^{1,0} \otimes \bigwedge^{2,0}$ and so $\xi \in [\![A]\!]$ corresponds to the condition that the manifold is almost-Kähler).

Specifically we wish to take as input an algebraic curvature tensor *Rend* written in block diagonal form w.r.t. a standard basis for $\bigwedge^2$.

```
eRend:=array(1..6,1..6,[
[-ea,-ewpf,0,0,0,0],
[-ewpf,-eb/2,-ewp00,0,0,0],
[0,-ewp00,-eb/2,0,0,0],
[0,0,0,-(ea+eb)/3,-ewm,0],
[0,0,0,-ewm,-(ea+eb)/3,0],
[0,0,0,0,0,-(ea+eb)/3]]);
```

$$eRend := \begin{bmatrix} -ea\,,\,-ewpf\,,\,0\,,\,0\,,\,0\,,\,0 \\ -ewpf\,,\,-\frac{1}{2}\,eb\,,\,-ewp00\,,\,0\,,\,0\,,\,0 \\ 0\,,\,-ewp00\,,\,-\frac{1}{2}\,eb\,,\,0\,,\,0\,,\,0 \\ 0\,,0\,,0\,,-\frac{1}{3}\,ea-\frac{1}{3}\,eb\,,-ewm\,,0 \\ 0\,,0\,,0\,,-ewm\,,-\frac{1}{3}\,ea-\frac{1}{3}\,eb\,,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,-\frac{1}{3}\,ea-\frac{1}{3}\,eb \end{bmatrix}$$

This sets up a curvature tensor in what we shall call the "Rend" notation — we have chosen it to be Einstein as this is the case we are interested in. However, the program in this section would work perfectly well even if it wasn't.

```
ebl2:=array(1..6,1..4,1..4,[
[[0,1,0,0],[-1,0,0,0],[0,0,0,1],[0,0,-1,0]],
```

```
[[0,0,1,0],[0,0,0,-1],[-1,0,0,0],[0,1,0,0]],
[[0,0,0,1],[0,0,1,0],[0,-1,0,0],[-1,0,0,0]],
[[0,1,0,0],[-1,0,0,0],[0,0,0,-1],[0,0,1,0]],
[[0,0,1,0],[0,0,0,1],[-1,0,0,0],[0,-1,0,0]],
[[0,0,0,1],[0,0,-1,0],[0,1,0,0],[-1,0,0,0]]]):
einstein(bl2[a,i,j] = 1/2*bl2[a,i,j]):
```

This gives a basis for $\bigwedge^2$, normalised appropriately.

```
einstein(R[i,j,k,l] = -bl2[a,i,j]*bl2[b,k,l]*Rend[a,b]):
```

This allows us to evaluate R from $Rend$.

We shall want some other, similar bases: `bl20`, a basis for $[\![\bigwedge^{2,0}]\!]$, `blM`, a basis for $\bigwedge^-$, `blP`, a basis for $\bigwedge^+$ and `bA` a basis for $[\![A]\!]$.

We know that three components of $\overline{\nabla}\xi$ are determined by R. These three components lie in the spaces $\mathcal{W}_F^+$, $\mathcal{W}_{00}^+$ and $\mathcal{R}_{00}$ and are the components of $\alpha\overline{\nabla}\xi$ where $\alpha : T^*M \otimes T^*M \otimes T^*M \longrightarrow \bigwedge^2 \otimes T^*M$ by antisymmetrisation on the first two factors.

We want to evaluate these components of $\overline{\nabla}\xi$ from R. Now if

$$tensor[A,B] \in \bigwedge^2 \otimes \bigwedge^{2,0}$$

then

```
bl20[A,k,l]*bA[alpha,j,A]*bA[alpha,b,B]*bl2[C,i,b]*tensor[C,B]
```

will lie in $T^*M \otimes [\![A]\!]$ by definition of bA. Hence by Schur's Lemma, if the component of $\alpha\overline{\nabla}\xi$ corresponding to W_F^+ is given by $tensor[A,B]$, then the corresponding component of $\overline{\nabla}\xi$ is given by the above formula up to a constant. Thus we only have to find the constant.

```
erpWPFagx:=array(1..6,1..2,[[1,0],[0,0],[0,0],[0,0],[0,0],[0,0]]):
```

We set up a tensor representing $\pi^{\mathcal{W}_{\mathcal{F}}^+}(\alpha(\overline{\nabla}\xi)$, viewed as an element of $\mathbb{C} \otimes \bigwedge^{2,0} \subseteq \bigwedge^2 \otimes \bigwedge^{2,0} \subseteq \mathrm{End}(\bigwedge^2)$ and written in block diagonal form w.r.t. the standard bases:

```
einstein(pWPFgx[i,j,k,l] = c* bl20[A,k,l]*bA[alpha,j,A]
*bA[alpha,b,B]*bl2[C,i,b]*rpWPFagx[C,B]):
```

So the actual projection of $\alpha(\overline{\nabla}\xi) \in T^*M \otimes T^*M \otimes T^*M \otimes T^*M$ should be given by the above formula. (r = readable, p = projection , P = plus, a = α, g = $\overline{\nabla}$, x = ξ is the code used in naming these tensors.)

We should have that:

```
einstein(rpWPFagx[A,B] =
bl2[A,i,j]*bl20[B,k,l]*1/2*pWPFgx[i,j,k,l]
-  bl2[A,i,j]*bl20[B,k,l]*1/2*pWPFgx[j,i,k,l]):
```

so let,

```
einstein(test[A,B] =  bl2[A,i,j]*bl20[B,k,l]*1/2*pWPFgx[i,j,k,l]
  -  bl2[A,i,j]*bl20[B,k,l]*1/2*p
WPFgx[j,i,k,l]):
print(eval(etest),eval(erpWPFagx));
```

$$\begin{bmatrix} ec & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus we conclude that c must be 1 so

```
pWPFgx[i,j,k,l]= bl20[A,k,l]*bA[alpha,j,A]
*bA[alpha,b,B]*bl2[C,i,b]*rpWPFagx[C,B]
```

We now wish to repeat the same procedure with W_{00}^{+} and R_{00} the end result is:

```
pWP00gx[i,j,k,l] = 1/2*bl20[A,k,l]*bA[alpha,j,A]
*bA[alpha,b,B]*bl2[C,i,b]*rpWP00agx[C,B]

pR00gx[i,j,k,l] = Ψbl20[A,k,l]*bA[alpha,j,A]
*bA[alpha,b,B]*bl2[C,i,b]*rpR00agx[C,B]
```

We would also like to be able to pick at random the remaining part of $\overline{\nabla}\xi$.

We first of all find a basis for $T^*M \otimes [\![A]\!]$ which we shall call bTA:

```
ebTA:=array(sparse,1..16,1..4,1..4,1..2):
for a from 1 to 4 do: for j from 1 to 4 do:
for k from 1 to 4 do: for l from 1 to 2 do:
ebTA[(j-1)*4 + a,a,k,l]:=ebA[j,k,l]:
od:od:od:od:
```

Now we shall let *alpha* be a 16 by 32 array which contains the information on how $gxi = \overline{\nabla}\xi$ gets mapped to $alphagxi = \alpha\overline{\nabla}\xi$ w.r.t. the basis bTA given above and a basis for for $TM * TM * \bigwedge^{2,0}$:

```
alpha:=array(sparse,1..16,1..32):
for a from 1 to 4 do: for j from 1 to 4 do:
```

```
for k from 1 to 4 do: for l from 1 to 2 do:
alpha[(j-1)*4 + a, 16*(l-1) + 4*(k-1) + a]:=
alpha[(j-1)*4 + a, 16*(l-1) + 4*(k-1) + a] + ebA[j,k,l]:
alpha[(j-1)*4 + a, 16*(l-1) + 4*(a-1) + k]:=
alpha[(j-1)*4 + a, 16*(l-1) + 4*(a-1) + k] - ebA[j,k,l]:
od:od:od:od:

alpha:=transpose(alpha):
term:=nullspace(alpha):
```

We shall now let $bremTA$ contain a basis for the parts of $T^*M \otimes [\![A]\!]$ which lie in the kernel of the antisymmetrisation map:

```
eterm:=array(1..6,1..16):
for i from 1 to 6 do: for j from 1 to 16 do:
eterm[i,j]:=op(i,term)[j]:
od:od:

i:='i': j:='j': k:='k': l:='l': a:='a':
alpha:='alpha': term:='term':

#einstein(bremTA[a,i,j,k,l] = term[a,alpha]*bTA[alpha,i,j,B]
*bl20[B,k,l]):
```

Thus bremTA[a,i,j,k,l] now contains a basis for the parts of $T^*M\otimes[\![A]\!]$ which lie in the kernel of antisymmetrisation on the first two factors.

So a typical $\overline{\nabla}\xi$ that lies in the kernel of alpha could be given by

```
ergx:=array(1..6,[eQ,0,0,0,0,0]):
einstein(gx[i,j,k,l] = rgx[a]*bremTA[a,i,j,k,l]):
```

So we now have that gx contains a tensor lying in the kernel of α.

We wish now to compute $gxi = \overline{\nabla}\xi$ from R as given above, s.t. its component in the kernel of alpha is that given above.

We deduce from our previous working that $pWPFgx$, $pWP00gx$ and $pR00gx$ (i.e. the components of $gx = \overline{\nabla}\xi$ in the various spaces) can be computed as follows::

```
einstein(rR[a,b] = - Rend[a,b]):

erpWPFagx:=array(sparse,1..6,1..2):
erpWPFagx[1,1]:=-1/2*erR[1,2]:
erpWPFagx[1,2]:=-1/2*erR[1,3]:

einstein( pWPFgx[i,j,k,l] =
bl20[A,k,l]*bA[alpha,j,A]*bA[alpha,b,B]*bl2[C,i,b]*rpWPFagx[C,B]):
```

```
erpR00agx:=array(sparse,1..6,1..2):
for i from 1 to 3 do: for j from 1 to 2 do:
erpR00agx[i,j]:=-1/2*erR[3+i,j+1]:
od:od:
i:='i': j:='j':

einstein(pR00gx[i,j,k,l] =
bl20[A,k,l]*bA[alpha,j,A]*bA[alpha,b,B]*bl2[C,i,b]*rpR00agx[C,B]):

erpWP00agx:=array(sparse,1..6,1..2):
erpWP00agx[2,2]:=-1/2*erR[2,3]:
erpWP00agx[3,1]:=-1/2*erR[3,2]:
trace:=erR[2,2] + erR[3,3]:
erpWP00agx[2,1]:=-1/2*(erR[2,2]-trace/2):
erpWP00agx[3,2]:=-1/2*(erR[3,3]-trace/2):

einstein(pWP00gx[i,j,k,l] =
1/2*bl20[A,k,l]*bA[alpha,j,A]*bA[alpha,b,B]
*bl2[C,i,b]*rpWP00agx[C,B]):
```

Thus $gx = \overline{\nabla}\xi$ is given by:

```
einstein(gx[i,j,k,l] = pWP00gx[i,j,k,l] + pWPFgx[i,j,k,l]
+ pR00gx[i,j,k,l] +gx[i,j,k,l]):
```

We wish now to test what we have done so far. We compute a tensor test as follows:

```
einstein(test[i,j,k,l] = R[i,j,k,l] + gx[i,j,k,l] - gx[j,i,k,l]):
einstein(test[a,b] = -bl2[a,i,j]*bl2[b,k,l]*test[i,j,k,l]):
```

We anticipate that it should be equal to the curvature tensor we started with except for the second and third columns which should be blank except two diagonal terms which should be equal.

```
eval(etest);
```

$$\begin{bmatrix} -ea\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0 \\ -ewpf\,,\,-\frac{1}{2}\,eb\,,\,0\,,\,0\,,\,0\,,\,0 \\ 0\,,\,0\,,\,-\frac{1}{2}\,eb\,,\,0\,,\,0\,,\,0 \\ 0\,,0\,,0\,,-\frac{1}{3}\,ea-\frac{1}{3}\,eb\,,-ewm\,,0 \\ 0\,,0\,,0\,,-ewm\,,-\frac{1}{3}\,ea-\frac{1}{3}\,eb\,,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,-\frac{1}{3}\,ea-\frac{1}{3}\,eb \end{bmatrix}$$

Exactly as it should be. We now wish to choose an appropriate ξ. We know that the only condition our lemma imposes on ξ is one on its norm. So we start by picking an arbitrary $x = \xi \in [\![A]\!]$:

```
erx:=array(1..4,[1,2,3,4]):
einstein(x[i,j,k]=rx[a]*bA[a,i,B]*bl20[B,j,k]):
```

We know that the square of its norm should be some multiple of $Rend[2,2]+Rend[3,3]$. So we normalise ξ appropriately including a constant c which we shall determine:

```
einstein(normsqx=x[i,j,k]*x[i,j,k]):
etrace:=erR[2,2]+erR[3,3]:
einstein(x[i,j,k]=c*sqrt(etrace)/sqrt(enormsqx)*x[i,j,k]):
```

Since we now have tensors representing ξ and $\overline{\nabla}\xi$, we can compute $\overline{R}$ as follows:

```
einstein(Rbar[i,j,k,l] = R[i,j,k,l] + gx[i,j,k,l] - gx[j,i,k,l]
-x[i,a,l]*x[j,k,a] + x[j,a,l]*x[i,k,a]
+x[a,k,l]*x[i,j,a] - x[a,k,l]*x[j,i,a]):
einstein(rRbar[A,B] = bl2[A,i,j]*bl2[B,k,l]*Rbar[i,j,k,l]):

eval(erRbar);
```

$$\begin{bmatrix}
ea + \frac{1}{2}\,ec^2\,eb\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0 \\
ewpf\,,\,\frac{1}{2}\,eb - \frac{1}{2}\,ec^2\,eb\,,\,0\,,\,0\,,\,0\,,\,0 \\
0\,,\,0\,,\,\frac{1}{2}\,eb - \frac{1}{2}\,ec^2\,eb\,,\,0\,,\,0\,,\,0 \\
-\frac{1}{3}\,ec^2\,eb\,,0\,,0\,,\,\frac{1}{3}\,ea + \frac{1}{3}\,eb\,,\,ewm\,,0 \\
\frac{1}{15}\,ec^2\,eb\,,\,0\,,\,0\,,\,ewm\,,\,\frac{1}{3}\,ea + \frac{1}{3}\,eb\,,\,0 \\
\frac{11}{30}\,ec^2\,eb\,,\,0\,,\,0\,,\,0\,,\,0\,,\,\frac{1}{3}\,ea + \frac{1}{3}\,eb
\end{bmatrix}$$

So in fact we must have that c=1. So we can take

```
einstein(x[i,j,k] =
sqrt(etrace)/sqrt(enormsqx)*rx[a]*bA[a,i,B]*bl20[B,j,k]):
einstein(Rbar[i,j,k,l] = R[i,j,k,l] + gx[i,j,k,l] - gx[j,i,k,l]
-x[i,a,l]*x[j,k,a] + x[j,a,l]*x[i,k,a]
+x[a,k,l]*x[i,j,a] - x[a,k,l]*x[j,i,a]):

einstein(Rend[a,b] = Rbar[i,j,k,l]*bl2[a,i,j]*bl2[b,k,l]):
eval(eRend);
```

$$\left[\begin{array}{cccccc} ea + \frac{1}{2}\,eb & 0 & 0 & 0 & 0 & 0 \\ ewpf & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3}\,eb & 0 & 0 & \frac{1}{3}\,ea + \frac{1}{3}\,eb & ewm & 0 \\ \frac{1}{15}\,eb & 0 & 0 & ewm & \frac{1}{3}\,ea + \frac{1}{3}\,eb & 0 \\ \frac{11}{30}\,eb & 0 & 0 & 0 & 0 & \frac{1}{3}\,ea + \frac{1}{3}\,eb \end{array}\right]$$

Thus we have now found tensors $x = \xi$ and $gx = \overline{\nabla}\xi$ which are compatible with the curvature R in the sense that they satisfy the lemma.

It will be helpful to compute the scalar and $*$-scalar curvature in terms of a and b. The formulae we use to compute them can be taken to be their definitions (they only agree with standard definitions up to scale).

```
eomega:=array(1..4,1..4,
[[0,1,0,0],[-1,0,0,0],[0,0,0,1],[0,0,-1,0]]):

einstein(starscalar=R[i,j,k,l]*omega[i,j]*omega[k,l]):
eval(estarscalar);
```

$$4\,ea$$

```
eid:=array(1..4,1..4,[[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]]):

einstein(scalar=R[i,j,k,l]*id[i,l]*id[j,k]):
eval(escalar);
```

$$-2\,ea - 2\,eb$$

```
einstein(normsqxi=x[i,j,k]*x[i,j,k]):
eval(enormsqxi);
```

$$eb$$

For future reference we write:

```
emu:=4: ev1:=-2: ev2:=-2: elambda:=1:
```

so that $R[k,l,alpha,beta]*omega[k,l]*omega[alpha,beta] = \mu a$, $s = v_1 * a + v_2 * b$ and $\|\xi\|^2 = \lambda b$.

A.3.2 Setting up values for $\overline{\nabla}\nabla\xi$ and ∇R

$\xi \in [\![\bigwedge^{1,0} \otimes \bigwedge^{2,0}]\!]$ so we can define η to be the part of ξ that lies in $\bigwedge^{1,0} \otimes \bigwedge^{2,0}$. So η is a $\bigwedge^{2,0}$ valued $(1,0)$-form. So we can define $\partial\eta$ to be a $\bigwedge^{2,0}$ valued $(2,0)$-form, and $\partial^*\eta$ to be a $\bigwedge^{2,0}$ valued $(0,0)$-form. The operators ∂ and ∂^* are defined using the Levi-Civita connection on the form part and $\overline{\nabla}$ on $\bigwedge^{2,0}$. However, since $\xi \otimes \xi$ has no components in common with either $\bigwedge^{2,0}$ or $\bigwedge^{2,0} \otimes \bigwedge^{2,0}$, we needn't worry about this.

If (M^4, g, J) is almost-Kähler and Einstein then $\overline{\nabla}R$ has two components isomorphic to $\langle u^3 \rangle \otimes V^-$. One comes from $\partial^*\partial\eta = \partial^* W^+_{00}$ and the other from $\partial\partial^*\eta = \partial W^+_F$. We shall in this section choose values for $\overline{\nabla}(\partial\eta)$ and $\overline{\nabla}(\partial^*\eta)$ and hence compute the $\langle u^3 \rangle \otimes V^-$ components of $\overline{\nabla}R$.

Firstly we know from Schur's Lemma that

$$\|\partial\eta\|^2 = c\|W^+_{00}\|^2$$
$$\|\partial^*\eta\|^2 = c\|W^+_F\|^2.$$

We wish to compute these coefficients. In order to do so we shall first need to set up a number of tensors and bases. We need a `BL10`, `BL01`, `BL20` and `BL02` which are bases for $\bigwedge^{1,0}$, $\bigwedge^{0,1}$, $\bigwedge^{2,0}$ and $\bigwedge^{0,2}$ respectively. We also define tensors `star0`, `star1`, ..., `star4` which represent the Hodge star acting on $\bigwedge^0$, $\bigwedge^1$ etc. (each of these has an appropriate normalisation).

Now suppose that $\xi \in [[\bigwedge^{1,0} \bigwedge^{2,0}]]$. We define η to be the component of ξ in $\bigwedge^{1,0} \bigwedge^{2,0}$.

```
einstein(eta[i,j,k]=BL10[A,i]*BL01[A,a]*x[a,j,k]):
```

Similarly we can calculate $\overline{\nabla}\eta$ — abbreviated to geta —

```
einstein(geta[a,i,j,k] = BL10[A,i]*BL01[A,b]*gx[a,b,j,k]):
```

So $\partial\eta$ is given as follows:

```
einstein(pdeta[a,b,j,k] = 2*BL20[a,b]*BL02[c,i]*geta[c,i,j,k]):
```

On the other hand $\partial^*\eta$ is given by:

```
einstein(pdstareta[j,k] =
-4*star4[a,b,c,d]*star1[b,c,d,i]*geta[a,i,j,k]):
```

The norms of these two tensors are related to the norms of W^+_{00} and W^+_F respectively. Note that we use the norms s.t. $\|e1 \wedge ... \wedge ep\|^2 = 1$ on the form components. This ensures that out formal adjoints are given by $d^* = - * d *$ exactly.

```
econjpdeta:=array(1..4,1..4,1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do:
for k from 1 to 4 do: for l from 1 to 4 do:
econjpdeta[i,j,k,l]:=conjugate(epdeta[i,j,k,l]):
```

```
od:od:od:od:
i:='i':j:='j':k:='k':l:='l':
einstein(normsqpdeta=1/2*pdeta[a,b,c,d]*conjpdeta[a,b,c,d]):
eval(simplify(enormsqpdeta));
```

$$\frac{1}{2}\, \mathit{ewp00}\, \mathrm{conjugate}(\, \mathit{ewp00}\,)$$

```
econjpdstareta:=array(1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do:
econjpdstareta[i,j]:=conjugate(epdstareta[i,j]):
od:od:
i:='i':j:='j':
einstein(normsqpdstareta=pdstareta[a,b]*conjpdstareta[a,b]):
eval(simplify(enormsqpdstareta));
```

$$\frac{1}{2}\, \mathit{ewpf}\, \mathrm{conjugate}(\, \mathit{ewpf}\,)$$

If we define tensors $T1$ and $T2$ lying in $T^*M \otimes \mathcal{W}^+_F$ and $T^*M \otimes \mathcal{W}^+_{00}$ respectively by:

```
eT1:=array(sparse,1..4,1..4,1..4,1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do:
for k from 1 to 4 do: for l from 1 to 4 do:
eT1[1,i,j,k,l]:=epWPFgx[i,j,k,l]/ewpf*eA:
od:od:od:od:

eT2:=array(sparse,1..4,1..4,1..4,1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do:
for k from 1 to 4 do: for l from 1 to 4 do:
eT2[1,i,j,k,l]:=epWP00gx[i,j,k,l]/ewp00*eB:
od:od:od:od:
```

(note that T1 is proportional to A and T2 is proportional to B) then we have that $ggx = \overline{\nabla\nabla}\xi = T1 + T2 +$ terms lying in other components. Thus as far as calculating $\partial\partial^*\eta$, $\partial^*\partial\eta$ is concerned we may as well take $ggx = T1 + T2$.

```
i:='i': j:='j': k:='k': l:='l':
assume(eA,real): assume(eB,real): assume(eb,real):

einstein(ggx[a,i,j,k,l]=T1[a,i,j,k,l] + T2[a,i,j,k,l]):
einstein(ggeta[a,i,j,k,l] = ggx[a,i,b,k,l]*BL01[A,b]*BL10[A,j]):
```

We can calculate from this what $\partial\partial^*\eta$ is:

```
einstein(pdpdstareta[B,j,k] =
-4*BL10[A,B]*BL01[A,alpha]*star4[a,b,c,d]
*star1[b,c,d,i]*ggeta[alpha,a,i,j,k]):
```

Let us define $strangeterm1 = (\partial\partial^*\eta, eta)$:

```
econjeta:=array(1..4,1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do: for k from 1 to 4 do:
econjeta[i,j,k]:=conjugate(eeta[i,j,k]):
od:od:od:
i:='i': j:='j': k:='k':

einstein(test=pdpdstareta[i,j,k]*conjeta[i,j,k]):

strangeterm1:=eval(simplify(etest));
```

$$strangeterm1 := -\frac{1}{60}\, I\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{1}{30}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

We can also calculate what $\partial^*\partial\eta$ is.

```
einstein(pdstarpdeta[l,j,k]=
-6*BL10[A,l]*BL01[A,B]*star3[B,alpha,beta,gamma]
*star2[beta,gamma,a,b]*BL20[a,b]*BL02[c,i]*ggeta[alpha,c,i,j,k]):
```

Let us define $strangeterm2 = (\partial^*\partial\eta, \eta)$:

```
einstein(test=pdstarpdeta[i,j,k]*conjeta[i,j,k]):

strangeterm2:=eval(simplify(etest));
```

$$strangeterm2 := -\frac{1}{20}\, I\, eB\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{1}{15}\, eB\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

Recall that the component of R in $\mathcal{W}_F^+$ is given by:

```
pWPFR[i,j,k,l]=-pWPFgx[i,j,k,l] + pWPF[j,i,k,l]
- pWPFgx[k,l,i,j] + pWPFgx[l,k,i,j]
```

and the component of R in $\mathcal{W}_{00}^+$ is given by:

```
pWP00R[i,j,k,l]=-pWP00gx[i,j,k,l] + pWP00gx[j,i,k,l]
```

Let S be the sum of these two components.

```
einstein(S[i,j,k,l] = -pWPFgx[i,j,k,l] + pWPFgx[j,i,k,l]
- pWPFgx[k,l,i,j] + pWPFgx[l,k,i,j] -
pWP00gx[i,j,k,l] + pWP00gx[j,i,k,l]):

einstein(rtest[a,b]=bl2[a,i,j]*bl2[b,k,l]*S[i,j,k,l]):
eval(ertest);
```

$$\begin{bmatrix} 0 & ewpf & 0 & 0 & 0 & 0 \\ ewpf & 0 & ewp00 & 0 & 0 & 0 \\ 0 & ewp00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as we expect. So we can determine $gS = \overline{\nabla}S$ from $T1$ and $T2$ as follows:

```
einstein(gS[a,i,j,k,l] =
-T1[a,i,j,k,l] + T1[a,j,i,k,l] - T1[a,k,l,i,j] + T1[a,l,k,i,j] -
T2[a,i,j,k,l] + T2[a,j,i,k,l]):
```

Now $\nabla R = \overline{\nabla}R - \xi R$ where ξR is computed as follows:

```
einstein(xR[a,i,j,k,l] =
x[a,b,i]*R[b,j,k,l] + x[a,b,j]*R[i,b,k,l]
+ x[a,b,k]*R[i,j,b,l] + x[a,b,l]*R[i,j,k,b]):
```

Thus as far as the component of ∇R in $\langle u^3\rangle \otimes V^-$ is concerned we can take $hR = \nabla R$ to be:

```
einstein(hR[a,i,j,k,l] = gS[a,i,j,k,l] - xR[a,i,j,k,l]):
```

One can check the working that we have done so far by reproving some well known results. For example one can easily check the coefficients in Sekigawa's integral formulae using the 2-jet we have constructed. The Chern–Weil theorem is proved using the differential Bianchi identity in a crucial way. Thus one can reprove the integral formula of Sekigawa by examining the differential Bianchi identity explicitly. One can check the entire 3-jet we have constructed by reproving Sekigawa's integral formula in this way.

A.3.3 Finding the linear relation between $\tau 1$, $\tau 2$ and $\tau 3$

Suppose that $T_{ijkl} \in T^* \otimes T^* \otimes \bigwedge^2$. Now $T^* \otimes T^* \otimes \bigwedge^2$ has exactly three components isomorphic to $S^2 V^+$. We call them $proj1$, $proj2$, $proj3$ and define them by:

```
proj1[A] = blP[A,k,l]*T[a,a,k,l]
```

```
proj2[B] = blP[B,i,l]*blM[A,alpha,l]*blM[A,a,b]*symm[i,alpha,a,b]
```

where

```
symm[i,j,k,l] = 1/2*T[i,j,k,l] + 1/2*T[j,i,k,l]
```

and

```
proj3[A] = blP[A,a,i]*Phi[a,j,k,l]*temporary[i,j,k,l]
```

where

```
asym[i,j,k,l]=1/2*T[i,j,k,l] - 1/2*T[j,i,k,l],
temporary[i,j,k,l] =
asym[i,j,k,l] + asym[i,k,l,j] + asym[i,l,j,k],
Phi:=array(antisymmetric,1..4,1..4,1..4,1..4):
Phi[1,2,3,4]:=1.
```

These correspond to the components found in the body of the paper. Suppose that T_{ijkl} satisfies $T_{i[jkl]} = 0$. Then since $T^*M \otimes \bigwedge^3$ contains a component isomorphic to S^2V^+ we must have $\alpha_1 proj1 + \alpha_2 proj2 + \alpha_3 proj3 = 0$ for some constants α_i. More generally, if we define

```
temporary[i,j,k,l] = T[i,j,k,l] + T[i,k,l,j] + T[i,l,j,k]
bT[A]=blP[A,a,i]*Phi[a,j,k,l]*temporary[i,j,k,l]
```

Then we should have $\alpha_1 proj1 + \alpha_2 proj2 + \alpha_3 proj3 = bT$. Thus to find these constants all we have to do is pick a random tensor $T \in T^* \otimes T^* \otimes \bigwedge^2$, compute $proj1$, $proj2$, $proj3$ and bT. Then if T is suitably generic, the equation $\alpha_1 proj1[A] + \alpha_2 proj2[A] + \alpha_3 proj3[A] = bT[A]$ should give us three equations in the three unknowns α_i which we shall then be able to solve. This is what we do in the next program.

```
_seed :=742284:
```

Changing the seed changes the random numbers generated.

Pick a tensor in $T^* \otimes T^* \otimes \bigwedge^2$

```
eT:=array(sparse,1..4,1..4,1..4,1..4):
for i from 1 to 4 do: for j from 1 to 4 do:
for k from 1 to 4 do: for l from k+1 to 4 do:
eT[i,j,k,l]:=rand(1..6)():
eT[i,j,l,k]:=-eT[i,j,k,l]:
od:od:od:od:

i:='i': j:='j': k:='k': l:='l':

einstein(proj1[A]=blP[A,k,l]*T[a,a,k,l]):
eval(eproj1);
```

$$[\,36\;1\;25\,]$$

```
einstein(symm[i,j,k,l]=1/2*T[i,j,k,l]+1/2*T[j,i,k,l]):

einstein(proj2[B]=
blP[B,i,l]*blM[A,alpha,l]*blM[A,a,b]*symm[i,alpha,a,b]):
eval(eproj2);
```

$$\left[\frac{5}{4}\;\frac{-1}{4}\;\frac{31}{4}\right]$$

```
ePhi:=array(antisymmetric,1..4,1..4,1..4,1..4):
ePhi[1,2,3,4]:=1:
einstein(asym[i,j,k,l]=1/2*T[i,j,k,l]-1/2*T[j,i,k,l]):
einstein(temporary[i,j,k,l]=
asym[i,j,k,l]+asym[i,k,l,j]+asym[i,l,j,k]):

einstein(proj3[A]=blP[A,a,i]*Phi[a,j,k,l]*temporary[i,j,k,l]):
eval(eproj3);
```

$$\left[\frac{-15}{2}\ -15\ 0\right]$$

So $proj1$, $proj2$, $proj3$ contain the values of each S^2V^+ component of $T \in T^* \otimes T^* \otimes \bigwedge^2$.

```
eA:=array(1..3,1..3):
for i from 1 to 3 do: for j from 1 to 3 do:
eA[i,j]:=eproj.i[j]:
od:od:
i:='i':j:='j':

rank(eA);
```

$$3$$

As one would expect, these terms are linearly independent.

Consider now $b : T^* \otimes T^* \otimes \bigwedge^2 \longrightarrow T^* \otimes \bigwedge^3 \longrightarrow T^* \otimes T^* \longrightarrow \bigwedge^+$:

```
einstein(temporary[i,j,k,l]=T[i,j,k,l]+T[i,k,l,j]+T[i,l,j,k]):
einstein(bT[A]=blP[A,a,i]*Phi[a,j,k,l]*temporary[i,j,k,l]):
eval(ebT);
```

$$[39\ -12\ -9]$$

```
constants:=linsolve(transpose(eA),ebT);
```

$$constants := \left[\frac{3}{2}\ -6\ 1\right]$$

```
einstein(test[A]=3/2*proj1[A] - 6*proj2[A] + 1*proj3[A]):
eval(etest);
```

$$[39\ -12\ -9]$$

This program proves that $3/2 * proj1 - 6 * proj2 + 1 * proj3 = 0$ for tensors T satisfying $T_{i[jkl]} = 0$.

A.3.4 Performing the calculation

We have that $\nabla\nabla R \in T^* \otimes T^* \otimes \bigwedge^2 \otimes \bigwedge^2$. Hence $\omega^{ab}(\nabla\nabla R)_{ijklab} \in T^* \otimes T^* \otimes \bigwedge^2$. But by the differential Bianchi identity, $\omega^{ab}(\nabla\nabla R)_{i[jkl]ab} = 0$. Thus from the previous program we see that if we define:

$$\begin{aligned}
proj1 &= \omega^{kl}\omega^{ab}(\nabla\nabla R)_{iiklab} \\
proj2 &= \omega^{il}(blM)_{A\alpha l}(blM)_{Aab} \times \\
&\quad \left(\frac{1}{2}(\nabla\nabla R)_{i\alpha ab\gamma\delta} + \frac{1}{2}(\nabla\nabla R)_{\alpha iab\gamma\delta}\right)\omega^{\gamma\delta} \\
proj3 &= \frac{1}{2}\omega^{ai}\Phi_{ajkl}\omega^{\gamma\delta} \times \\
&\quad ((\nabla\nabla R)_{ijkl\gamma\delta} + (\nabla\nabla R)_{iklj\gamma\delta} + (\nabla\nabla R)_{iljk\gamma\delta} \\
&\quad - (\nabla\nabla R)_{jikl\gamma\delta} - (\nabla\nabla R)_{kjil\gamma\delta} - (\nabla\nabla R)_{ikjl\gamma\delta})
\end{aligned}$$

then we shall have $\frac{3}{2}proj1 - 6proj2 + proj3 = 0$. We wish therefore to evaluate these terms.

By the Ricci identity,

$$\begin{aligned}
(\nabla\nabla R)_{ijkl\gamma\delta} - (\nabla\nabla R)_{jikl\gamma\delta} &= R_{ijak}R_{al\gamma\delta} + R_{ijal}R_{ka\gamma\delta} \\
&\quad + R_{ija\gamma}R_{kla\delta} + R_{ija\delta}R_{kl\gamma a}.
\end{aligned}$$

So if we define

$$\begin{aligned}
temp[i,j,k,l] &= (R_{ijak}R_{al\gamma\delta} + R_{ijal}R_{ka\gamma\delta} \\
&\quad + R_{ija\gamma}R_{kla\delta} + R_{ija\delta}R_{kl\gamma a})\omega^{\gamma\delta},
\end{aligned}$$

then we have that

$$\begin{aligned}
proj3 &= \frac{1}{2}\omega^{ai}\Phi_{ajkl} \times \\
&\quad (temp[i,j,k,l] + temp[i,k,l,j] + temp[i,l,j,k]).
\end{aligned}$$

We can evaluate this with the following program:

```
einstein(T[i,j,k,l]=R[i,j,a,k]*R[a,l,alpha,beta]*omega[alpha,beta]
+R[i,j,a,l]*R[k,a,alpha,beta]*omega[alpha,beta]
+R[i,j,a,alpha]*R[k,l,a,beta]*omega[alpha,beta]
+R[i,j,a,beta]*R[k,l,alpha,a]*omega[alpha,beta]):

einstein(wp3 =
  1/2*omega[a,i]*Phi[a,j,k,l]*T[i,j,k,l]
+ 1/2*omega[a,i]*Phi[a,j,k,l]*T[i,k,l,j]
+ 1/2*omega[a,i]*Phi[a,j,k,l]*T[i,l,j,k]):
eval(simplify(ewp3));
```

$$-6\, ewpf^2 + 6\, ea\, eb\tilde{} + 12\, ewp00^2 - 3\, eb\tilde{}^2$$

Hence

$$proj3 = 12\|W_{00}^+\|^2 - 6\|W_F^+\|^2 + 3(2a-b)b.$$

We now turn our attention to evaluating $proj2$.

$$\begin{aligned} proj2 &= \frac{1}{2}(blM)_{A\alpha l}(blM)_{Aab} \times \\ &\quad ((\nabla\nabla R)_{i\alpha ab\gamma\delta}\omega^{\gamma\delta} + (\nabla\nabla R)_{\alpha iab\gamma\delta}\omega^{\gamma\delta}) \end{aligned}$$

But since $R_F \equiv ((blM)_{Aab}R_{ab\gamma\delta}\omega^{\gamma\delta})_{i\alpha} \equiv 0$ we see that:

$$\begin{aligned} 0 &= \nabla\nabla((blM)_{Aab}R_{ab\gamma\delta}\omega^{\gamma\delta})_{i\alpha} \\ &= (blM)_{Aab}(\nabla_i((\nabla_\alpha\omega^{\gamma\delta})R_{ab\gamma\delta} + \omega^{\gamma\delta}\nabla_\alpha R_{ab\gamma\delta}) \\ &= (blM)_{Aab}(\nabla\nabla\omega)^{\gamma\delta}_{i\alpha}R_{ab\gamma\delta} \\ &\quad + (blM)_{Aab}(\nabla_\alpha\omega^{\gamma\delta})\nabla_i R_{ab\gamma\delta} \\ &\quad + (blM)_{Aab}(\nabla_i\omega^{\gamma\delta})(\nabla_\alpha R_{ab\gamma\delta} \\ &\quad + (blm)_{Aab}\omega^{\gamma\delta}(\nabla\nabla R)_{i\alpha ab\gamma\delta} \end{aligned}$$

Thus if we write $hJ = \nabla J = \nabla\omega$ and $hhJ = \nabla\nabla J$ then we have:

$$\begin{aligned} proj2 &= -\frac{1}{2}[\omega^{il}(blM)_{A\alpha l}(blM)_{Aab}(hhJ)_{i\alpha\gamma\delta}R_{ab\gamma\delta} \\ &\quad + \omega^{il}(blM)_{A\alpha l}(blM)_{Aab}(hhJ)_{\alpha i\gamma\delta}R_{ab\gamma\delta} \\ &\quad + 2\omega^{il}(blM)_{A\alpha l}(blM)_{Aab}(hJ)_{\alpha\gamma\delta}(hR)_{iab\gamma\delta} \\ &\quad + 2\omega^{il}(blM)_{A\alpha l}(blM)_{Aab}(hJ)_{i\gamma\delta}(hR)_{\alpha ab\gamma\delta}] \end{aligned}$$

Before we can evaluate this, we need to initialise values for hJ and hhJ. We proceed as follows: Recall that $\xi = -\frac{1}{2}J\nabla J$. In other words:

$$J_{ak}\xi_{ija} = \frac{1}{2}(\nabla_i J)_{jk}.$$

So we must set:

$$(hJ)_{ijk} = 2J_{ak}\xi_{ija}.$$

Differentiating the first formula gives:

$$2(\nabla_{alpha})\xi_{ija} + 2J_{ak}(\nabla_\alpha\xi)_{ija} = (\nabla\nabla J)_{\alpha ijk}$$

Hence:

$$4J_{bk}\xi_{\alpha ab}\xi_{ija} + 2J_{ak}(\nabla_\alpha\xi)_{ija} = (\nabla\nabla J)_{\alpha ijk}$$

Since $\nabla = \overline{\nabla} - \xi$, we have that:

$$\begin{aligned} (\nabla_\alpha\xi)_i ja &= (\overline{\nabla}_\alpha\xi)_{ija} \\ &= -\xi_{\alpha bi}\xi_{bja} - \xi_{\alpha bj}\xi_{iba} - \xi_{\alpha ba}\xi_{ijb}. \end{aligned}$$

We conclude that:

$$\begin{aligned}(hhJ)_{\alpha ijk} &= 4J_{bk}\xi_{\alpha ab}\xi_{ija} + 2J_{ak}(\overline{\nabla}_\alpha \xi)_{ija} \\ &\quad - 2J_{ak}\xi_{\alpha bi}\xi_{bja} - 2J_{ak}\xi_{\alpha bj}\xi_{iba} - 2J_{ak}\xi_{\alpha ba}\xi_{ijb}\end{aligned}$$

So we first of all make these assignments:

```
eJ:=eomega:

einstein(hJ[i,j,k]=2*J[a,k]*x[i,j,a]):
einstein(hhJ[alpha,i,j,k]=
4*J[b,k]*x[alpha,a,b]*x[i,j,a] + 2*J[a,k]*gx[alpha,i,j,a]
-2*J[a,k]*x[alpha,b,i]*x[b,j,a]
-2*J[a,k]*x[alpha,b,j]*x[i,b,a]
-2*J[a,k]*x[alpha,b,a]*x[i,j,b]):
```

We can now compute $proj2$:

```
einstein(wp2 =
-1/2*omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hhJ[i,alpha,gamma,delta]
          *R[a,b,gamma,delta]
-1/2*omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hhJ[alpha,i,gamma,delta]
          *R[a,b,gamma,delta]
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ[alpha,gamma,delta]
          *hR[i,a,b,gamma,delta]
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ[i,gamma,delta]
          *hR[alpha,a,b,gamma,delta]):

eval(simplify(ewp2));
```

$$0$$

We expect that $proj2$ can be written as a sum of squares of norms of curvature terms plus possibly a term ab and also terms which are complex multiples of $(\partial\partial^*\eta, \eta)$ and $(\partial^*\partial\eta, \eta)$. We can be confident by Schur's Lemma that the curvature terms vanish. To make absolutely sure that the other terms vanish, we separate out the effect of each complex component of ∇J on the above term and check that they each vanish:

```
einstein(hJ10[i,j,k]=2*J[a,k]*eta[i,j,a]):
einstein(hJ01[i,j,k]=2*J[a,k]*conjeta[i,j,a]):

einstein(wp2t1 =
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ01[alpha,gamma,delta]
            *gS[i,a,b,gamma,delta]
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ01[i,gamma,delta]
            *gS[alpha,a,b,gamma,delta]):
```

```
eval(simplify(ewp2t1));
```

$$0$$

```
einstein(wp2t2 =
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ10[alpha,gamma,delta]
            *gS[i,a,b,gamma,delta]
-omega[i,l]*blM[A,alpha,l]*blM[A,a,b]*hJ10[i,gamma,delta]
            *gS[alpha,a,b,gamma,delta]):

eval(simplify(ewp2t2));
```

$$0$$

We conclude that $proj2 = 0$.

Finally we consider $proj1$.

$$proj1 = \omega^{kl}\omega^{ab}(\nabla\nabla R)_{iiklab}$$

Now,

$$\begin{aligned}
(\nabla\nabla(\omega^{kl}\omega^{ab}R_{klab}))_{ii} &= \nabla_i((\nabla_i\omega^{kl})\omega^{ab}R_{klab} \\
&\quad + \omega^{kl}(\nabla_i\omega^{ab})R_{klab} + \omega^{kl}\omega^{ab}(\nabla_i R_{klab})) \\
&= (\nabla\nabla\omega^{kl})_{ii}\omega^{ab}R_{klab} \\
&\quad + (\nabla_i\omega^{kl})(\nabla_i\omega^{ab})R_{klab} \\
&\quad + (\nabla_i\omega^{kl})\omega^{ab}(\nabla_i R_{klab}) \\
&\quad + (\nabla_i\omega^{kl})(\nabla_i\omega^{ab})R_{klab} \\
&\quad + \omega^{kl}(\nabla\nabla\omega^{ab})_{ii}R_{klab} \\
&\quad + \omega^{kl}(\nabla_i\omega^{ab})(\nabla_i R_{klab}) \\
&\quad + (\nabla_i\omega^{kl})\omega^{ab}(\nabla_i R_{klab}) \\
&\quad + \omega^{kl}(\nabla_i\omega^{ab})(\nabla_i R_{klab}) \\
&\quad + \omega^{kl}\omega^{ab}(\nabla_i\nabla_i R_{klab}).
\end{aligned}$$

Hence

$$\begin{aligned}
proj1 &= \Delta(\omega^{kl}\omega^{ab}R_{klab}) \\
&\quad - 2(\nabla_i\omega^{kl})(\nabla_i\omega^{ab})R_{klab} \\
&\quad - 2\omega^{kl}(\nabla_i\omega^{ab})(\nabla_i R_{klab}) \\
&\quad - 2(\nabla_i\omega^{kl})\omega^{ab}(\nabla_i R_{klab}) \\
&\quad -(\nabla_i\nabla_i\omega^{kl})\omega^{ab}R_{klab} \\
&\quad -\omega^{kl}(\nabla_i\nabla_i\omega^{ab})R_{klab} \\
&= proj1term1 + proj1term2
\end{aligned}$$

where

$$\begin{aligned} proj1term2 \quad = \quad & -2(hJ)_{ikl}(hJ)_{iab}R_{klab} \\ & -2\omega^{kl}(hJ)_{iab}(hR)_{iklab} \\ & -2(hJ)_{ikl}\omega^{ab}(hR)_{iklab} \\ & -(hhJ)_{iikl}\omega_{ab}R_{klab} \\ & -\omega_{kl}(hhJ)_{iiab}R_{klab} \end{aligned}$$

and where $proj1term1 = \Delta(\omega^{kl}\omega^{ab}R_{klab}) = \Delta(\mu a)$. Since $s = v_1 a + v_2 b$,

$$v_1\Delta a + v_2\Delta b = 0.$$

Since $\|\xi\|^2 = \lambda b$ we have that:

$$proj1term1 = \mu\Delta a = -\frac{\mu v_2}{\lambda v_1}\Delta\|\xi\|^2.$$

We evaluate $proj1term2$:

```
einstein(proj1term2 =
-2*hJ[i,k,l]*hJ[i,a,b]*R[k,l,a,b]
-2*omega[k,l]*hJ[i,a,b]*hR[i,k,l,a,b]
-2*hJ[i,k,l]*omega[a,b]*hR[i,k,l,a,b]
-hhJ[i,i,k,l]*omega[a,b]*R[k,l,a,b]
-omega[k,l]*hhJ[i,i,a,b]*R[k,l,a,b]):

eval(simplify(eproj1term2));
```

$$-8\, ewpf^2 + 4\, eb\tilde{}^{\,2} + \frac{16}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} - 8\, ea\, eb\tilde{}$$

Once again this expression is only accurate for the first three terms. We perform a similar refinement.

```
einstein(proj1term2t1 =
-2*omega[k,l]*hJ01[i,a,b]*gS[i,k,l,a,b]
-2*hJ01[i,k,l]*omega[a,b]*gS[i,k,l,a,b]):
eval(simplify(eproj1term2t1));
```

$$-\frac{4}{15}\, I\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{8}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

```
eval(strangeterm1*16);
```

$$-\frac{4}{15}\, I\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{8}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

```
einstein(proj1term2t2 =
-2*omega[k,l]*hJ10[i,a,b]*gS[i,k,l,a,b]
-2*hJ10[i,k,l]*omega[a,b]*gS[i,k,l,a,b]):
eval(simplify(eproj1term2t2));
```

$$\frac{4}{15}\, I\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{8}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

```
eval(conjugate(strangeterm1)*16);
```

$$\frac{4}{15}\, I\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15} + \frac{8}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

```
eval(-8*ewpf^2 -4*(2*ea-eb)*eb
+ 16*strangeterm1 + 16*(conjugate(strangeterm1)));
```

$$-8\, ewpf^2 - 4\,(\,2\, ea - eb\tilde{}\,)\, eb\tilde{} + \frac{16}{15}\, eA\tilde{}\, \sqrt{eb\tilde{}}\, \sqrt{15}$$

So we have that

$$proj1term2 = -8\|W_F^+\|^2 - 4*(2a-b)*b + (16\partial\partial^* eta, eta) + (16\overline{\partial\partial}^*\eta, \overline{\eta}).$$

```
proj3:=simplify(ewp3);
proj2:=0:
 proj1term2:= -8*ewpf^2 + 4*eb^2 - 8*ea*eb
+ 16*pdpdstaretaeta+ 16*pdbpdbstaretabetab;
```

$$proj3 := -6\, ewpf^2 + 12\, ewp00^2 - 3\, eb\tilde{}^2 + 6\, ea\, eb\tilde{}$$

$$proj1term2 := -8\, ewpf^2 + 4\, eb\tilde{}^2 - 8\, ea\, eb\tilde{} + 16\, pdpdstaretaeta \\ + 16\, pdbpdbstaretabetab$$

```
proj1term1:=-emu*ev2/(elambda*ev1)*laplacianxisq;
```

$$proj1term1 := -4\, laplacianxisq$$

```
proj1:=proj1term1+proj1term2:

3/2*proj1 - 6*proj2 + proj3;
```

$$-6\, laplacianxisq - 18\, ewpf^2 + 3\, eb\tilde{}^2 - 6\, ea\, eb\tilde{} + 24\, pdpdstaretaeta \\ + 24\, pdbpdbstaretabetab + 12\, ewp00^2$$

So

$$-6\Delta\|xi\|^2 - 18\|W_F^+\|^2 - 3(2a-b)b + 12\|W_{00}^+\|^2 + 24(\partial\partial^*\eta,\eta) + 24(\overline{\partial\partial}^*\overline{\eta},\overline{\eta}) = 0$$

at every point — this proves Proposition 4.2.1. Integrating this gives:

$$\int -18\|W_F^+\|^2 + 12\|W_{00}^+\|^2 + 48\|\partial^*\eta\|^2 - 3(2a-b)b = 0$$

. Recall that $\|\partial^*\eta\|^2 = 1/2\|W_F^+\|^2$.

```
pdpdstaretaeta:=enormsqpdstareta: assume(ewpf,real):
pdbpdbstaretabetab:=enormsqpdstareta:

simplify(3/2*proj1 - 6*proj2 + proj3);
```

$$-6\, laplacianxisq + 3\, eb^{\sim 2} - 6\, ea\, eb^{\sim} + 6\, ewpf^{\sim 2} + 12\, ewp00^2$$

as claimed.

Bibliography

[Ada69] J. F. Adams. *Lectures on Lie Groups.* W. A. Benjamin, Inc., New York, 1969.

[AG97] V. Apostolov and P. Gauduchon. The Riemannian Goldberg–Sachs theorem. *Internat. J. Math*, 8(4):421–439, 1997.

[AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London*, 362:425–461, 1978.

[Apo97] V. Apostolov. *Le Tenseur de Weyl d'une Surface Complexe Hermitienne.* PhD thesis, Ecole Polytechnique, Palaiseau, France, 1997.

[AS68] M. F. Atiyah and I. M. Singer. The index of elliptic operators III. *Ann. Math.*, 87:546–604, 1968.

[Ati67] M. F. Atiyah. *K-Theory.* Benjamin, New York, 1967.

[BCG+91] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior Differential Systems.* Number 18 in Mathematical Sciences Research Institute Publications. Spring–Verlag, New York, 1991.

[BCG94] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Inv. Math.*, 118, 1994.

[Ber82] L. Bérard Bergery. Sur de nouvelles variétés Riemanniennes d'Einstein. *Publications de l'Institut E. Cartan*, 4:1–60, 1982.

[Bes87] A. L. Besse. *Einstein Manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1987.

[BF82] C. P. Boyer and J. D. Finley. Killing vectors in self-dual Euclidean Einstein spaces. *J. Math. Phys.*, pages 1126–30, 1982.

[BI86] D. E. Blair and S. Ianus. Critical associated metrics on symplectic manifolds. *Contemp. Math*, 51:23–29, 1986.

[Bla90] D. E. Blair. Nonexistence of 4-dimensional almost Kähler manifolds of constant curvature. *Proc. Amer. Math. Soc.*, 110(4), Dec 1990.

[BP81] J. P. Bourguignon and A. Polombo. Integrands des nombres caractéristiques et coubure: Rien ne va plus dès la dimension 6. *J. Diff. Geom.*, 16:537–550, 1981.

[Cal82] E. Calabi. Extremal kähler metrics. In S.-T. Yau, editor, *Seminar on Differential Geometry*. Princeton, 1982.

[Car45] E. Cartan. *Les systems diffèrentielles extérieurs et leurs applications géométriques*. Hermann, 1945.

[Der83] A. Derdzinski. Self-dual kähler manifolds and einstein manifolds of dimension 4. *Comp. Math.*, 49:405–433, 1983.

[DF89] S. K. Donaldson and R. Friedman. Connected sums of self-dual manifolds and deformations of singular spaces. *Nonlinearity*, 2:197–239, 1989.

[DK81] D. DeTurck and J. Kazdan. Some regularity theorems in Riemannian geometry. *Ann. Scient. Ec. Norm. Sup. 4 série*, pages 249–260, 1981.

[DK90] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1990.

[DM90] J. Davidov and O. Muskarov. Twistor spaces with Hermitian Ricci tensor. *Proc. Amer. Math. Soc.*, 109(4):1115–1120, 1990.

[Don83] S. K. Donaldson. An application of gauge theory to the topology of four-manifolds. *J. Diff. Geom.*, 18:269–315, 1983.

[Don90] S. K. Donaldson. Polynomial invariants for smooth four-manifolds. *Topology*, 29(3):257–315, 1990.

[Don96] S. K. Donaldson. The Seiberg–Witten equations and 4-manifold topology. *Bull. Amer. Math. Soc. (N. S.)*, 33(1):45–70, 1996.

[Dra97] T. C. Draghici. *Special Metrics on Symplectic Manifolds*. PhD thesis, Michigan State University, 1997.

[EGS62] L. Ephrenpreis, V. W. Gullemin, and S. Sternberg. Deformations of structures on manifolds defined by transitive, continuous pseudogroups. *Ann. of Math*, 76:306–445, 1962.

[FFS94] M. Falcitelli, A. Farinola, and S. Salamon. Almost-Hermitian geometry. *Diff. Geom. and Appl.*, 4:259–282, 1994.

[Flo91] A. Floer. Self-dual conformal structures on lCP^2. *J. Diff. Geom.*, 33:551–373, 1991.

[Gas82] J. Gasqui. Sur la résolubilité locale des équations d'Einstein. *Compositio Mathematica*, 47:43–69, 1982.

[GH78] G. W. Gibbons and S. W. Hawking. Gravitational multi-instantons. *Phys. Lett.*, 78(2):430–432, 1978.

[GH80] A. Gray and L. Hervella. The sixteen classes of almost Hermitian manifolds and their invariants. *Ann. Mat. Pura Appl.*, 123:35–58, 1980.

[GM94] G. Grantcharov and O. Muskarov. Hermitian $*$-Einstein surfaces. *Proc. Amer. Math. Soc.*, 120:233–239, 1994.

[Gol67a] H. Goldschmidt. Existence theorems for linear partial differential equations. *Ann. of Math.*, 86(247-270), 1967.

[Gol67b] H. Goldschmidt. Integrability criteria for systems of non-linear partial differential equations. *J. Differential Geom.*, 1:269–307, 1967.

[Gol69] S. I. Goldberg. Integrability of almost Kähler manifolds. *Proc. Amer. Math. Soc.*, 21:96–100, 1969.

[Gol72] H. Goldschmidt. Sur la structure des équations de Lie I. le troisième théorème fondamental. *J. Differential Geom.*, 6:367–373, 1972.

[Gra70] A. Gray. Nearly Kähler manifolds. *J. Diff. Geom.*, 4:283–309, 1970.

[Gro82] M. Gromov. Volume and bounded cohomology. *Publ. Math. I.H.E.S.*, 56:5–99, 1982.

[GS64] V. Guillemin and S. Sternberg. An algebraic model of transitive differential geometry. *Bull. Amer. Math. Soc.*, 70:16–47, 1964.

[Her60] R. Hermann. A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. *Proc. AMS*, 11:236–242, 1960.

[Hit74] N. Hitchin. On compact four-dimensional Einstein manifolds. *J. Diff. Geom*, 9:435–442, 1974.

[Hit75] N. Hitchin. On the curvature of rational surfaces. In *"Differential Geometry", Proc. of symp. Pure Math*, volume 27, pages 65–80, 1975.

[Hit81] N. J. Hitchin. Kählerian twistor spaces. *Proc. London Math. Soc.*, 43:133–150, 1981.

[HKLR87] N. Hitchin, A. Karlhede, U. Lindström, and M. Rocek. Hyperkähler metrics and supersymmetry. *Comm. Math. Phys.*, 108, 535-589 1987.

[KK92] A. D. King and D. Kotschick. The deformation theory of anti-self-dual conformal structures. *Math. Ann.*, 294:591–609, 1992.

[KM94] P. Kronheimer and T. Mrowka. The genus of embedded surfaces in the projective plane. *Math. Res. Lett.*, 1:215–222, 1994.

[KN63] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry I.* Interscience. Wiley, New York, 1963.

[Kob95] P. Kobak. Explicit doubly Hermitian 4-manifolds. *ESI preprint*, 1995.

[KP95] J. Kim and M. Pontecorvo. A new method of constructing scalar-flat Kähler surfaces. *J. Diff. Geom.*, 41:449–477, 1995.

[Kur62] M. Kuranishi. *Lectures on Exterior Differential Systems.* Tata Institute of Fundamental Research, Bombay, 1962.

[LeB91a] C. LeBrun. Scalar-flat Kähler metrics on blown-up ruled surfaces. *J. Reine Angew. Math*, 420:223–253, 1991.

[LeB91b] C. R. LeBrun. Explicit self-dual metrics on $CP_2\#\ldots\#CP_2$. *J. Diff. Geom.*, 34:223–53, 1991.

[LeB95a] C. LeBrun. Einstein metrics and Mostow rigidity. *Math. Res. Lett.*, 2:1–8, 1995.

[LeB95b] C. LeBrun. Polarized 4-manifolds, extremal Kähler metrics and Seiberg–Witten theory. *Math. Res. Lett.*, 2(5):653–662, 1995.

[LeB96] C. LeBrun. Four manifolds without Einstein metrics. *Math. Res. Lett.*, 3(2):133–147, 1996.

[LeB97] C. LeBrun. Einstein metrics on complex surfaces. In *Geometry and Physics*, volume 184 of *Lecture notes in pure and applied mathematics*, pages 167–177, 1997.

[LL95] T. J. Li and A. Liu. General wall crossing formula. *Math. Res. Lett.*, 2(6):797–810, 1995.

[LM96] F. Lalonde and D. McDuff. J-curves and the classification of rational and ruled symplectic 4-manifolds. In C. B. Thomas, editor, *Contact and Symplectic Geometry*, Publications of the Newton Institute. Cambridge University Press, 1996.

[LS93] C. LeBrun and M. A. Singer. Existence and deformation theory for scalar-flat Kähler surfaces. *Invent. Math.*, 112(2):273–313, 1993.

[Mor96] J. W. Morgan. *The Seiberg–Witten equations and Applications to the Topology of Smooth Four-Manifolds*, volume 44. Princeton University Press, Princeton, New Jersey, 1996.

[MS95] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Mathematical Monographs. OUP, 1995.

[MW96] L. J. Mason and N. M. J. Woodhouse. *Integrability, self-duality and twistor theory.* The Clarendon Press, Oxford University Press, 1996.

[Nur] P. Nurowski. *Einstein Equations and Cauchy–Riemann Geometry.* PhD thesis, SISSA.

[Ogu97] T. Oguro. On almost Kähler manifolds of constant curvature. *Tsukuba J. Math.*, 21(1):199–206, 1997.

[Ols78] Z. Olszak. A note on almost Kähler manifolds. *Bull. Acad. Polon. Sci.*, XXVI:139–141, 1978.

[OO96a] H. Ohta and K. Ono. Note on symplectic manifolds with $b^+ = 1$, ii. *Internat. J. Math.*, 7(6):775–770, 1996.

[OO96b] H. Ohta and K. Ono. Notes on symplectic manifolds with $b^+ = 1$. In M. Maruyama, editor, *Moduli of Vector Bundles*, volume 179, pages 209–212. Dekker, 1996.

[OO97] H. Ohta and K. Ono. Symplectic 4-manifolds with $b_2^+ = 1$. In *Geometry and Physics (Aarhus 1995)*, volume 184 of *Lecture Notes in Pure and Appl. Math*, pages 237–244, 1997.

[Pag79] D. Page. A compact rotating gravitational instanton. *Phys. Lett.*, 79:235–238, 1979.

[PB83] M. Przanowski and B. Broda. Locally Kähler gravitational instantons. *Acta Phys. Pol. B*, 14:637–661, 1983.

[PB87] M. Przanowski and S. Bialecki. Lie–bäcklund transformations and gravitational instantons. *Acta Phys. Polon. B*, 18(10):879–889, 1987.

[Pen72] R. Penrose. *Techniques of differential topology in relativity.* SIAM, 1972.

[Pen76] R. Penrose. Nonlinear gravitons and curved twistor theory. *Gen. Relativ. Grav.*, I:31–52, 1976.

[PN] M. Przanowski and P. Nurowski. An example of a Ricci-flat almost-Kähler manifold. ESI preprint.

[PP91] H. Pedersen and Y. S. Poon. Hamiltonian constructions of Kähler Einstein metrics and Kähler metrics of constant scalar curvature. *Commun. Math. Phys.*, 136:309–26, 1991.

[PR86a] R. Penrose and W. Rindler. *Spinors and Space-Time.* Cambridge University Press, 1986.

[PR86b] R. Penrose and W. Rindler. *Spinors and Space-Time*, volume 2. Cambridge University Press, 1986.

[Pro88] M. H. Protter. Maximum principles. In P. W. Schaeffer, editor, *Maximum Principles and Eigenvalue Problems in Partial Differential Equations*, volume 175 of *Pitman Research Notes in Mathematics Series*, 1988.

[Prz91] M. Przanowski. Killing vector fields in self-dual, Euclidean Einstein spaces with $\lambda \neq 0$. *J. Math. Phys.*, 32:1004–1010, 1991.

[Qui64] D. G. Quillen. *Formal Properties of Over-Determined Systems of Linear Partial Differential Equations.* PhD thesis, Harvard University, 1964.

[Sal82] S. Salamon. Topics in four-dimensional Riemannian geometry. In E. Vesentini, editor, *Geometry Seminar "Luigi Bianchi"*, number 1022 in Lecture Notes in Mathematics. Springer-Verlag, 1982.

[Sal84] S. Salamon. Harmonic and holomorphic maps. In E. Vesentini, editor, *Geometry Seminar "Luigi Bianchi" II*, number 1164 in Lecture Notes in Mathematics. Springer-Verlag, 1984.

[Sal89] S. Salamon. *Riemannian Geometry and Holonomy Groups.* Pitman Research Notes in Mathematics. Longman, 1989.

[Sal91] S. Salamon. Special structures on four-manifolds. *Riv. Mat. Univ. Parma*, 17(4):109–123, 1991.

[Sek85] K. Sekigawa. On some 4-dimension compact Einstein almost Kähler manifolds. *Math. Ann.*, 271:333–337, 1985.

[Sek87] K. Sekigawa. On some compact Einstein almost Kähler manifolds. *J. Math. Soc. Japan*, 39(4):677–684, 1987.

[SO94] K. Sekigawa and T. Oguro. Non-existence of almost Kähler structure on hyperbolic spaces of dimension $2n \geq 4$. *Math. Ann.*, 300(2):317–329, 1994.

[SP94] A. Swann and H. Pedersen. The Hitchin–Thorpe inequality for Einstein–Weyl manifolds. *Bull. London Math. Soc.*, 26(2):191–194, 1994.

[ST69] I. M. Singer and J. A. Thorpe. The curvature of 4-dimensional Einstein spaces. In *"Global Analysis", Papers in Honour of K. Kodaira*, pages 355–365. Princeton University Press, 1969.

[SV90] K. Sekigawa and L. Vanhecke. Four-dimensional almost Kähler Einstein manifolds. *Ann. Mat. Pura Appl.*, CLVII:149–160, 1990.

[Swe67] W. J. Sweeney. The δ-Poincaré estimate. *Pacific J. Math*, 20:559–570, 1967.

[Tau92] C. H. Taubes. The existence of anti-self-dual conformal structures. *J. Diff. Geom.*, 36:163–253, 1992.

[Tau94] C. H. Taubes. The Seiberg–Witten invariants and symplectic forms. *Math. Res. Lett.*, 1:809–822, 1994.

[Tau95a] C. H. Taubes. More constraints on symplectic manifolds from Seiberg–Witten equations. *Math. Res. Lett.*, 2:9–14, 1995.

[Tau95b] C. H. Taubes. The Seiberg–Witten and the Gromov invariants. *Math. Res. Lett.*, 2:221–238, 1995.

[Tau96] C. H. Taubes. From the Seiberg–Witten equation to pseudo holomorphic curves. *J. Amer. Math. Soc.*, 9:845–918, 1996.

[Tho69] J. A. Thorpe. Some remarks on the Gauss–Bonnet integral. *J. Math. Mec.*, 18:779–786, 1969.

[Tia87] G. Tian. On Kähler-einstein metrics on certain Kähler manifolds with $c_1(m) > 0$. *Invent. Math.*, 89:225–246, 1987.

[Tod97a] K. P. Tod. private communication, 1997.

[Tod97b] K. P. Tod. The $SU(\infty)$-Toda field equation and special four-dimensional metrics. In *Geometry and Physics (Aarhus 1995)*, volume 184 of *Lecture Notes in Pure and Appl. Math*, pages 167–176, 1997.

[TV91] F. Tricerri and L. Vanhecke. Curvature tensors on almost Hermitian manifolds. *Trans. Amer. Math. Soc.*, 267(2):365–398, Oct 1991.

[Vil70] J. Vilms. Totally geodesic maps. *J. Diff. Geom.*, 4:73–79, 1970.

[Wit94] E. Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1:764–796, 1994.

[Wu48] W. T. Wu. Sur la structure presque complexe d'une variété différentiable réelle de dimension 4. *C.R.A.S.*, 227:1076–1078, 1948.