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2025.11.06

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KCL Pensions25

Collective Pensions with Investment Choice

Collective Pensions with Investment Choice

Two-year research project funded by Nuffield

- UK context - fully-funded pension with opt out and potentially with employer contributions
- Study existing/proposed UK collective pensions (shared-indexation designs)
- Compare with a tontine-based approach (collective drawdown = advice + tontine)
- Study the mathematical limits of collective pensions

Credits to: James Dalby, Rohan Hobbs, Catherine Donnelly, Cristin Buescu, Pension Policy Institute, Advisory Board

Decoupling investment risk and longevity risk

Name	Age	Asset growth	Pot (start year)	Pot (year emd)	Prob dying	Contribution
Alice	70	4%	£200,000	£208,000	2%	£4160
Bob	80	-2%	£150.000	£147,000	6%	£8820
Cyril	100	2%	£10,000	£10.200	36%	

Cyril dies leaving £10,200

- Alice receives £3,265
- Bob receives £6,935

"Collective drawdown" is our name for the combination of this tontine structure with sensible investment/consumption advice.

No mutually beneficial contracts

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Required assumptions

- Preferences define an ordering on outcomes
- Infinite risk is unacceptable
- Additional money is always strictly better
- A finite time-horizon
- Preferences depend only on your own experience

Optimality of collective-drawdown

Corollary

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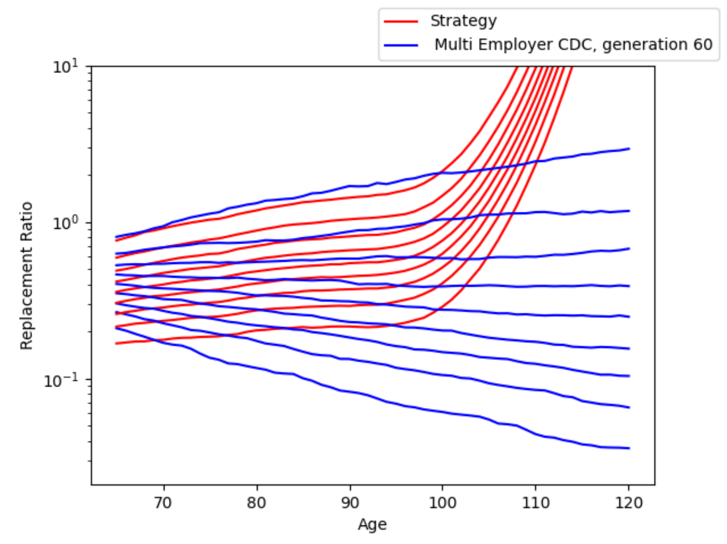
Proof:

The hard bit is proving that the Black-Scholes model is a complete market, but this is well known.

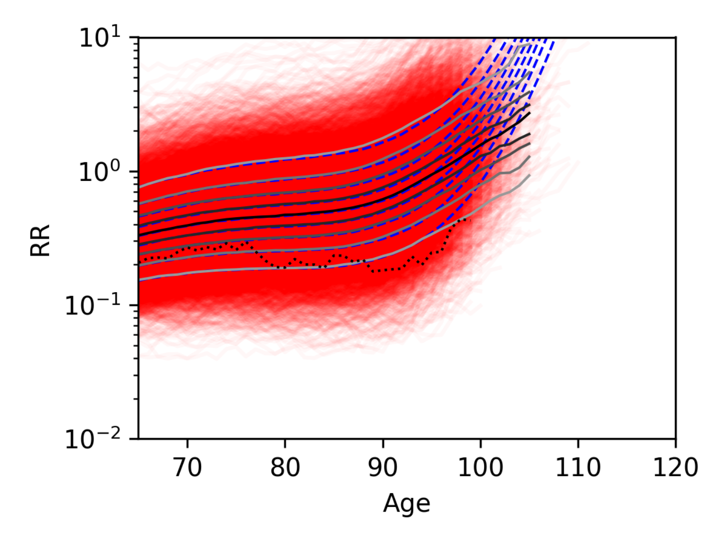
- There are no mutually beneficial contracts between infinite collectives of identical individuals.
- The optimal pension obtained for a group of identical individuals is an upper bound on the pensions available in a complete market
- This upper bound can be approximated very well using a tontine of disparate individuals.

Collective drawdown vs shared-indexation

Assuming infinite identical individuals, optimal collective drawdown strategy found using machine learning (details later...)

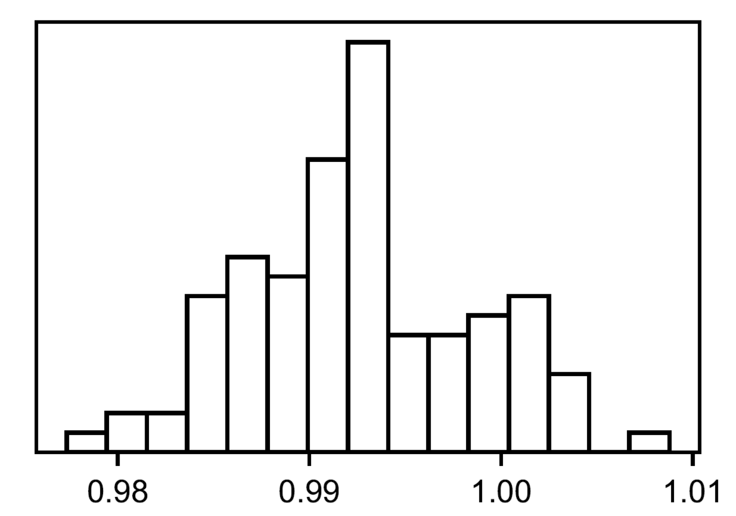


Finite fund size (20 per generation)

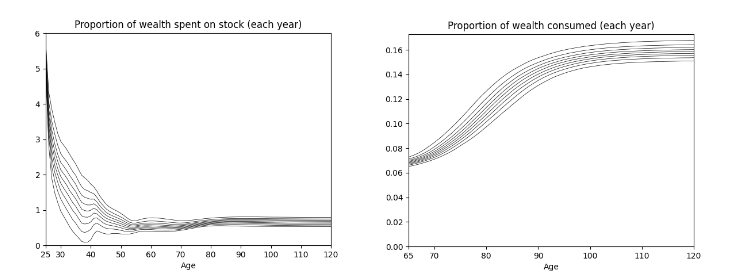


Finite funds

The “optimality ratio” for a single fund of 100 individuals with varying mortality and varying preferences



Optimal investment and consumption

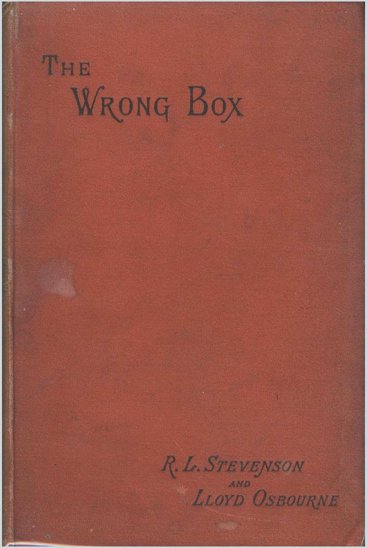


- The optimal strategy is highly leverageed
- This reflects the high leverage in shared-indexation designs
- The fund as a whole is not highly leveraged
- Imposing some restriction on maximum leverage makes a minimal difference

Other advantages

- Collective drawdown is very easy to understand
- Collective drawdown funds do not need to be large
- Scalability determined by transaction costs etc.
- Investment pooling for leverage is important
- Allows investment choice
- Allows easy buy-in and buy-out (subject to underwriting)

Problem – tontines are (possibly) illegal in the UK

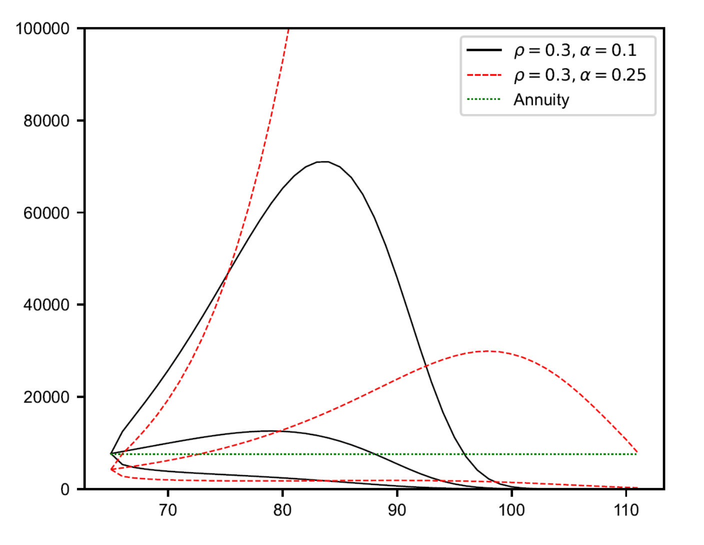


Providing investment advice

We have selected investment advice using optimization

- Merton suggested using an expected utility
- Asset pricing puzzles suggest we need to separate “satiation” and “risk”. Satiation often called elasticity of intertemporal substitution.
- Homogeneous Epstein-Zin preferences give analytically tractable formulae, including for collective drawdown

Sample Epstein-Zin outcomes (unsatisfactory examples)

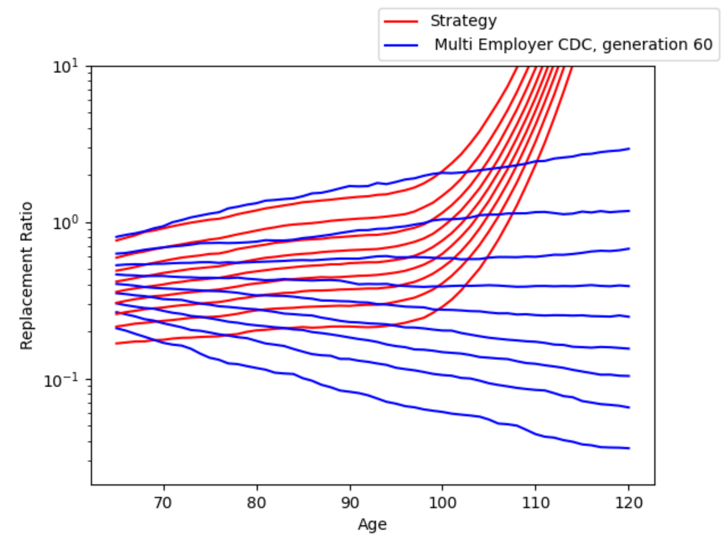


Our philosophy

- Choose a large, parsimonious family of preferences
- Solve by machine learning
- Examine outcomes to determine your preferences
- Include satiation, risk and adequacy
- Test the machine learning is close to optimal using classical solution methods
- Validate the machine learning algorithm with simulations

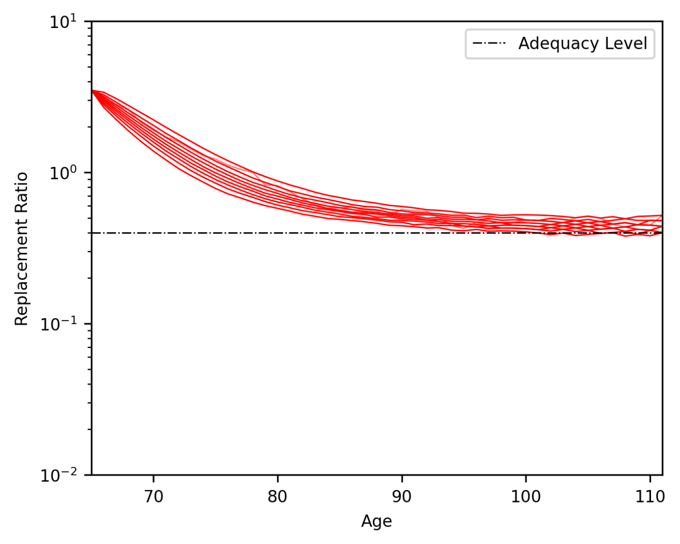
$$\text{gain} = \mathbb{E}\left(-\lambda \sum_{t\leq\tau}\left(\frac{c_t^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}\right)\right)$$

Choosing a utility function



A decumulation-only strategy

This illustrates the strategy in decumulation of an individual with high risk-aversion and a low, achievable, adequacy level.



Systematic longevity risk

- So far we've been assuming no systematic longevity risk
- We've also studied optimal investment with two models for systematic longevity risk
- Model 1: Highly stylised with a time symmetry. Yields analytic solutions

$$d\lambda_t = a\lambda_t^2 dt + b\lambda^3 2dW_t$$

- Model 2: A one-factor approximation to the Cairns-Blake-Dowd model. Realistically calibrated.
- Estimate effect of systematic longevity risk is $\pm 6\%$

Mutual insurance

Fundamental principle

- Complete the market by allowing insurance in additional risk factors
- Determine the price of insurance by market clearance

We can solve using PDE in the case when one type of investor dominates the market

- 1-factor Cairns-Blake-Dowd model or stylised model
- All investors same age
- Investors have Epsten-Zin preferences
- Analytic solution for stylised model

Results

	$\alpha_1 = -10, \rho_1 = -1$	$\alpha_1 = -5, \rho_1 = -1$	$\alpha_1 = -3, \rho_1 = -1$	$\alpha_1 = -2, \rho_1 = -1$	$\alpha_1 = 3/20, \rho_1 = 1/3$	$\alpha_1 = 1/4, \rho_1 = 1/3$
$\alpha_2 = -10, \rho_2 = -1$	0%	7.76%	22.2%	37.6%	623%	6196%
$\alpha_2 = -5, \rho_2 = -1$	4.96%	0%	1.93%	5.48%	47.4%	92%
$\alpha_2 = -3, \rho_2 = -1$	10.2%	1.41%	0%	0.62%	14%	22.7%
$\alpha_2 = -2, \rho_2 = -1$	13.6%	3.22%	0.5%	0%	5.72%	8.42%
$\alpha_2 = 3/20$ $\rho_2 = 1/3$	21.8%	8.89%	4.32%	2.29%	0%	0.065%
$\alpha_2 = 1/4$ $\rho_2 = 1/3$	21.1%	8.32%	3.87%	1.93%	0.041%	0%

More advanced examples

What we can't do.,.

- Solve the ODE when there are two finitely sized groups of individuals!
- Solve the problem by PDE methods in this case (but we could have tried harder...)
- Solve the problem by machine-learning

What we can do...

- Solve a 1-period problem by machine-learning....

Conclusion

- Collective drawdown without additional insurance is probably close to optimal in realistic models
- The challenges of mutual insurance are likely to outweigh the benefits

Group	Design	Certainty Equivalent
Personal	DC + Annuity	35%
	DC + Flex then Fix	51%
Collective – shared-indexation	Flat-accrual CDC (e.g. Royal Mail)	≤44%
	Dynamic-accrual CDC (Multiemployer)	≤45%
	Statistically calibrated CDC	≤52%
Collective	Collective Drawdown	62%